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Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 3, 579--589
Persistant URL: http://dml.cz/dmlcz/105802

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A NOTE ON BANACH SPACES WITHOUT THE APPROXIMATION PROPERTY

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Abstract: We show that a certain class of Banach sequence spaces has closed subspaces without the approximation property. Certain Lorentz sequence spaces provide particularly interesting examples.

Key words: Banach space, approximation property, Lorentz sequence space, Banach sequence space.

AMS: Primary 46B15 Ref. Z.: 7.972.22
Secondary 46A45

Introduction. Grothendieck [4] asked whether every Banach space has the approximation property (a.p.). Enflo [3] showed that there is a closed subspace of the Banach space $c_0$ which fails to have a.p. Since then further papers have appeared giving examples of Banach spaces without a.p. One of the most notable is the paper by Davie [2]. He shows that not only $c_0$ but also the $\ell_p$ spaces ($2 < p < \infty$) have closed subspaces without a.p. The purpose of this note is to show that some simple modifications of Davie's elegant construction can be used to show that a certain class of Banach sequence spaces all have closed subspaces which fail to have a.p. We will mention a variety of spaces to which this result can be applied but we will focus attention on a
class of Lorentz sequence spaces.

Let \( w = (w_n) \) be a nonincreasing sequence of positive numbers with \( \lim w_n = 0 \) and \( \sum w_n = \infty \). For \( 1 \leq p < \infty \), let \( d(w, p) \) denote the space of all complex sequences \( a = (a_n) \) for which \( \|a\| = \sup \left( \sum |a_{\pi(n)}| \right)^{1/p} < \infty \), the supremum being taken over all permutations \( \pi \) of the positive integers. The space \( d(w, p) \) endowed with the norm \( \|\cdot\| \) is a Banach space called a Lorentz sequence space. For information on Lorentz sequence spaces see [1, 5].

The spaces \( d(w, p) \) resemble in certain respects the \( \ell_p \) spaces; in particular, they share with the \( \ell_p \)-spaces the property that each of their infinite dimensional closed subspaces contains a closed subspace isomorphic to \( \ell_p [5; p. 30] \). In light of this and since the question as to whether \( \ell_p (1 \leq p < 2) \) has a closed subspace without a.p. is unsettled and apparently difficult, one might guess that it would be hard to decide whether or not \( d(w, p) (1 \leq p < 2) \) has a closed subspace without a.p. However, we will see that if the weighting sequence \( (w_n) \) satisfies an appropriate growth condition, then it is an easy consequence of our result that \( d(w, p) \) has a closed subspace \( E(w, p) \) without a.p. There is another interesting aspect of the examples \( E(w, p) (1 \leq p \leq 2) \). When examples of Banach spaces without a.p. have been constructed, no consideration seems to have been given as to whether or not these were isomorphic to the examples already constructed by Davie. It will be easy to see that the spaces \( E(w, p) (1 \leq p \leq 2) \) are not isomorphic to any of the Davie spaces.
2. Davie's Construction for Certain Banach Sequence Spaces. Let X be a countably infinite set. Let $M^+$ denote the set of nonnegative functions on X with values in the extended reals. A map $\varphi$ on $M^+$ to the extended reals is a function norm if $\varphi$ satisfies the following for all $f, g$ in $M^+$:

(i) $\varphi(f) \geq 0$ and $\varphi(f) = 0$ if and only if $f = 0$,
(ii) $\varphi(af) = a\varphi(f)$ for all constants $a \geq 0$,
(iii) $\varphi(f + g) \leq \varphi(f) + \varphi(g)$,
(iv) $f \leq g$ implies $\varphi(f) \leq \varphi(g)$.

We extend the definition of $\varphi$ to $M$, the set of all complex-valued functions on X by $\varphi(f) = \varphi(|f|)$. Let $\ell^p \varphi$ denote the set of all $f$ in $M$ satisfying $\varphi(f) < \infty$. It is well known [7; p. 444] that $(\ell^p \varphi, \varphi)$ is a Banach space if and only if $\varphi$ has the property that for every sequence $(f_n)$ of nonnegative functions in $\ell^p \varphi$ satisfying $\sum \varphi(f_n) < \infty$, we have $\varphi(\sum f_n) = \sum \varphi(f_n)$. We will refer to such a Banach space $(\ell^p \varphi, \varphi)$ as a Banach sequence space. These spaces are special cases of the "Banach function spaces" studied extensively by Luxemburg and Zaanen and others (see [7; Chapter 15]).

We now modify Davie's construction to fit the setting of a Banach sequence space. The probabilistic steps in Davie's proof need no modification and we will refer the reader to [2] for these steps.

Let $(\ell^p \varphi, \varphi)$ be a Banach sequence space. Let $(G_k)_{k=0}^\infty$ be a sequence of disjoint subsets of X with the cardinality of $G_k$ being $3 \cdot 2^k$. Define $H_k = G_{k-1} \cup G_k \cup G_{k+1}$ for $k \geq 0$ where we take $G_{-1} = \varnothing$. Let $G = \bigcup_{k=0}^\infty G_k$. 

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Lemma. If there is \( \eta > 0 \) such that \( \eta < \mathcal{F}(\chi_{\mathcal{I}G}) < \infty \) for all \( g \) in \( G \), then \( \| u \|_{\infty} < \eta^{-1} \mathcal{F}(u) \) for all \( u \) in \( L^{\infty} \) supported by \( G \).

Proof. Take \( u \) in \( L^{\infty} \) supported by \( G \). Let \( g \) belong to \( G \). Then \( \mathcal{F}(u) \geq \mathcal{F}(\| u(g) \| \chi_{\mathcal{I}G}) = \| u(g) \| \mathcal{F}(\chi_{\mathcal{I}G}) \geq \| u(g) \| \eta \).

Theorem 1. Suppose that for \( \eta > 0 \), \( \eta < \mathcal{F}(\chi_{\mathcal{I}G}) < \infty \) for all \( g \) in \( G \). Further suppose there is \( 2 < r \leq \infty \) and a constant \( C \) such that

\[
(1) \quad \mathcal{F}(\chi_{H_k}) \leq C \| \chi_{H_k} \|_r
\]

for all \( k \geq 0 \). Then there is a closed subspace of \( L^r \) without a.p.

Remark. Condition (1) is the key. Even for rather complicated norms \( \mathcal{F} \) it seems to be easy to check as it involves only the easily dealt with functions \( \chi_{H_k} \).

Proof. Regard the \( G_k \)'s as cyclic groups. It is shown in [2] via a probabilistic argument that the \( 3 \cdot 2^k \) characters on \( G_k \) can be partitioned into \( \sigma_1^k, \ldots, \sigma_{2^k}^k \) and \( \tau_1^k, \ldots, \tau_{2^{k+1}}^k \) so that

\[
(2) \quad |2 \sum_{j=1}^{2^k} \sigma_j^k(g) - 2^{k+1} \sum_{j=1}^{2^{k+1}} \tau_j^k(g)| \leq A_2 (k + 1)^{1/2} 2^{k/2}
\]

for some constant \( A_2 \) and all \( g \) in \( G_k \). Let \( \epsilon_j^k = \pm 1 \) for \( k \geq 0, 1 \leq j \leq 2^k \). (For now the choice of \( \epsilon_j^k \) will not matter. Later we will need a choice that will yield inequality (11) below.)

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Define functions $e^k_j (k \geq 0, 1 \leq j \leq 2^k)$ by

\[
\begin{align*}
e^k_j(g) = \begin{cases} 
  \varepsilon_{j-1}^k(g), & \text{if } g \in G_{k-1} \text{ and } k \geq 1 \\
  \varepsilon_{j}^k \sigma_j^{k}(g), & \text{if } g \in G_k \\
  0, & \text{otherwise.}
\end{cases}
\end{align*}
\]

Then $(e^k_j) \subset L_p$ since each $e^k_j$ has finite support in $G$ and $\varphi(\chi_{\{g\}}) < \infty$ for all $g$ in $G$. Let $E$ be the closed span of $(e^k_j)$.

Define the complex linear functionals

\[
\begin{align*}
\alpha^k_j (k \geq 0, 1 \leq j \leq 2^k) \text{ by}
\end{align*}
\]

\[
\alpha^k_j(f) = 3^{1.2^{-k}} \sum_{g \in G_k} \varepsilon_{j}^k \sigma_j^{k}(g^{-1})f(g)
\]

for all $f$ in $E$. Then $(\alpha^k_j) \subset E^*$, the dual of $E$. We see this by taking $(f_n)$ and $f$ in $E$ such that $\varphi(f_n - f) \rightarrow 0$. Then $|\alpha^k_j(f_n - f)| \leq \max_{g \in G_k} |f_n(g) - f(g)| \rightarrow 0$ as $n \rightarrow \infty$ since

\[
|f_n(g) - f(g)| \cdot \varphi(\chi_{\{g\}}) \leq \varphi(f_n - f) \text{ for all } g \text{ in } G.
\]

Similarly one can see that the right side of (5) below is in $E^*$.

Further $\alpha^k_j(e^k_1) = \sigma^k_1 \cdot \sigma^k_{k\ell}$ in (4) and (4) agrees with the right side of (5) on $(e^k_1 : k \geq 0, 1 \leq 1 \leq 2^\ell)$. Therefore since both (4) and the right side of (5) are continuous and linear, we have for $k \geq 1$ and $f$ in $E$

\[
\alpha^k_j(f) = 3^{1.2^{-k}} \sum_{g \in G_{k-1}} \varepsilon_{j-1}^k(g^{-1})f(g). \tag{5}
\]

Define linear functionals $\beta^k_k (k \geq 0)$ on the bounded linear operators from $E$ to $E$ by
\[ \beta^k(T) = 2^{-k} \sum_{j=1}^{2^k} \alpha_j(T e_j^k). \]

Since \( \alpha_j^k(e_j^k) = 1 \), we have \( \beta^k(I) = 1 \) where \( I \) is the identity operator. Using (4) we obtain

\[ \beta^k(T) = 3^{-1} 4^{-k} \sum_{\varphi \in G_k} T(\sum_{j=1}^{2^k} \tau_j^k(g^{-1})e_j^k)(g), \]

and using (5) with \( k + 1 \) instead of \( k \),

\[ \beta^{k+1}(T) = 6^{-1} 4^{-k} \sum_{\varphi \in G_k} T(\sum_{j=1}^{2^{k+1}} \tau_j^{k+1}(g^{-1})e_j^{k+1})(g). \]

(6) and (7) yield

\[ \beta^{k+1}(T) - \beta^k(T) = 3^{-1} 2^{-k} \sum_{\varphi \in G_k} T(\phi_g^k)(g), \]

where

\[ \phi_g^k = 2^{-k-1} \sum_{j=1}^{2^{k+1}} \tau_j^{k+1}(g^{-1})e_j^{k+1} - 2^{-k} \sum_{j=1}^{2^k} \tau_j^k(g^{-1})e_j^k. \]

Then the support of \( \phi_g^k \) is in \( H_k \) and \( \phi_g^k \) is in \( E \) for all \( g \) in \( G_k \). From (8) and the Lemma we see that

\[ |\beta^{k+1}(T) - \beta^k(T)| \leq \sup_{\varphi \in G_k} \| T\phi_g^k \|_\infty \leq \gamma^{-1} \sup_{\varphi \in G_k} \gamma(T\phi_g^k). \]

Davie shows via a probabilistic argument that \( \varepsilon_j^k \) can be chosen in (3) above so that

\[ |\phi_g^k(h)| \leq A_3(k + 1)^{1/2} 2^{-k/2} \]

for \( h \) in \( H_k \) and \( g \) in \( G_k \). From our hypothesis there is
$2 < r \leq \infty$ and a constant $A_4$ such that

\begin{equation}
\varphi(\phi_{g}^k) \leq A_4(k + 1)^{1/2} 2^{-k/2} \| \mathcal{H}_{k} \|_{r^*}.
\end{equation}

Now define $K = \{0 \} \cup \{ e_1^0 \} \cup \{(k + 1)^2 \phi_{g}^k : k \geq 0, \ g \ in \ G_k \} \subset E$. By (11), $K$ is compact in $L_\varphi$ and by (9),

$$|\beta^{k+1}(T) - \beta^k(T)| \leq \eta^{-1}(k + 1)^{-2} \sup_{x \ in \ K} \varphi(Tx)$$

for all bounded linear operators $T$ from $E$ to $E$. Further

$$|\beta^0(T)| \leq \eta^{-1} \sup_{x \ in \ K} \varphi(Tx).$$

Hence, for all bounded linear operators $T$ from $E$ to $E$, $\beta(T) = \lim_{k \to \infty} \beta^k(T)$ exists and satisfies

\begin{equation}
|\beta(T)| \leq 3\eta^{-1} \sup_{x \ in \ K} \varphi(Tx).
\end{equation}

Then since $\beta^k(I) = 1$ for all $k$,

\begin{equation}
\beta(I) = 1.
\end{equation}

Now $S_n \to S$ uniformly on $K$ in $L_\varphi$ implies by (12) that

$$\lim \ |\beta(S_n) - \beta(S)| = 0.$$

Now if $Tx = \varphi(x)e_1^k, \ \varphi \ in \ E^*$, then $\beta^\ell(T) = 0$ for $\ell > k$ and so $\beta(T) = 0$. By the linearity of $\beta$, $\beta(T) = 0$ when $Tx = \varphi(x)z$ for $\varphi$ in $E^*$ and $z$ in $\text{span}(e_1^k)$. Now for $Tx = \varphi(x)z$ with $\varphi$ in $E^*$ and $z$ in $E$ take $(z_n) \subset \text{span}(e_1^k)$ such that $\varphi(z_n - z) \to 0$. Let $T_n x = \varphi(x)z_n$. Then

$$\| T_n - T \| = \sup_{\varphi(x) \equiv 1} \varphi(T_n x - Tx) \leq \sup_{\varphi(x) \equiv 1} |\varphi(x)| \varphi(z_n - z) \to 0 \text{ as } n \to \infty.$$

Therefore $\beta(T_n) \to \beta(T)$, so $\beta(T) = 0$. By the linearity of $\beta$, $\beta(T) = 0$ for all finite rank operators $T$ from $E$ to $E$. It now follows from (13) and the sen-
tence following (13) that there cannot be a sequence of finite rank operators \((T_n)\) from \(E\) to \(E\) such that \(T_n \rightarrow I\) uniformly on \(K\). Therefore \(E\) fails to have a.p.

3. Examples. Since the above is a simple modification of Davie's argument it is not surprising that Davie's result should follow easily from it. Take \((G_k)\) to be a sequence of disjoint subsets of the positive integers of the proper cardinality. For \(2 < p < \infty\), let \(\varphi\) be the \(\ell_p\)-norm and take \(r = p\) and \(C = 1\). Theorem 1 implies that there is a closed subspace of \(\ell_p\), call it \(E_p\), without a.p. Alternately let \(\varphi\) be the \(\ell_\infty\) norm except take \(\varphi = \infty\) for any sequence which fails to converge to 0. Then \(\ell_\varphi\) is a Banach sequence space \([7; p. 449, Exercise 65.2]\); in fact \((\ell_\varphi, \varphi) = (c_0, \| \cdot \|_\infty)\). In this case taking \(r = \infty\) and \(C = 1\) and applying the theorem, one gets a closed subspace \(E_\infty\) of \(c_0\) without a.p.

Remark. The subspace \(E_p\) is not uniquely determined by \(p\). In particular any choice of the \((\varepsilon_{k,j})\) for which inequality (10) holds produces a closed subspace, \(E_p(\varepsilon_{k,j})\), without a.p. One may think of choosing the sequence \((\varepsilon_{k,j})\) of \(\pm 1\)'s at random. If this is done, one may ask for the probability of obtaining a subspace without a.p. It is an easy consequence of the Borel-Cantelli Lemma \([6; p. 70]\) that the probability is one.

Theorem 2. Take \((G_k)\) to be a sequence of disjoint subsets of the positive integers of proper cardinality and let \(1 \leq p < \infty\). Let \(d(w,p)\) be a Lorentz sequence space consisting of sequences supported by \(G\). Suppose there exists \(r > 2\) and
a number $M$ such that

\[(\sum_{n=1}^{\infty} w_n)^{1/p} \leq M k^{1/r}, \quad k = 1, 2, \ldots .\]

Then $d(w,p)$ has a closed subspace $E(w,p)$ without a.p. In particular, $d((1/n),p)$ has a closed subspace $E((1/n),p)$ without a.p. for all $p$, $1 \leq p < \infty$.

**Proof.** Let $\phi$ be the Lorentz norm. For $g$ in $G$,

$\phi(\chi_{\{g\}}) = w_1^{1/p} > 0$. Also by (14)

$\phi(\chi_{H_k}) \leq M(3 \cdot 2^{k-1} + 3 \cdot 2^k + 3 \cdot 2^{k+1})^{1/r} = M \|\chi_{H_k}\|_r$. Hence $d(w,p)$ has a closed subspace without a.p.

The fact that $d((1/n),p)$ has a closed subspace without a.p. for all $p$ $(1 \leq p < \infty)$ follows from the fact that

$$\lim_{k \to \infty} \left( \sum_{n=1}^{k} \frac{1}{n} - \ln k \right) = \gamma$$

Euler's constant.

Given the Davie space $E_p$ ($2 \leq p \leq \infty$) it is easy to generate further spaces without a.p.; indeed $E_p \oplus B$ is such a space for any Banach space $B$. It is natural to ask if our spaces are distinct from these.

**Theorem 3.** Let $1 \leq p \leq 2$ and let $w$ satisfy the hypotheses of Theorem 2. Then $d(w,p)$ is nonisomorphic to $E_p \oplus B$ for any choice of $r$ $(2 \leq r \leq \infty)$ and any choice of a Banach space $B$.

**Proof.** Fix an appropriate $d(w,p)$. Let $r$ and $B$ be given. Suppose $d(w,p)$ were isomorphic to $E_p \oplus B$. $E_p$, being a closed infinite-dimensional subspace of $\ell_p$, has a closed subspace isomorphic to $\ell_p$ [5; p. 30]. But then $d(w,p)$ would have a closed subspace $F$ isomorphic to $\ell_p$. But $F$, being a closed subspace of $d(w,p)$, would have a further closed sub-
space isomorphic to $\ell_p$. Thus $\ell_r \cong F$ has $\ell_p$ as a closed subspace where $r \neq p$. But it is well known that this cannot happen [5; p. 31].

We finish by briefly mentioning some further examples that can be obtained from Theorem 1. Let $(p_j)_{j=0}^{\infty}$ be given so that $1 < p_0 < \infty$ and $2 < r \leq p_j < \infty$ for $j = 1, 2, \ldots$. The Banach space $A = \left( \bigoplus_{j=1}^{\infty} \ell_{p_j} \bigoplus_{p_0} \right)$ can be regarded as a Banach sequence space where $X$ is taken as the set of pairs of positive integers. Subspaces $E$ of $A$ without a.p. can be constructed in a variety of ways. Unlike the earlier examples the choice of the sequence $(G_k)$ affects the nature of the subspace. As one example it is possible to take the $G_k$'s so that $E$ contains closed subspaces isomorphic to each of $\ell_{p_j}$, $j = 0, 1, 2, \ldots$. Such an $E$ is clearly not isomorphic to any of the Davie spaces $F_p$ nor to any of the spaces $E(\omega, p)$ from Theorem 2. Theorem 1 may also be applied to certain of the Orlicz sequence spaces.

References


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(Oblatum 1.6. 1977)