Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 4, 623--637

Persistent URL: http://dml.cz/dmlcz/105807

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

18,4 (1977)

INDUCTIVE DIMENSIONS FOR COMPLETELY REGULAR SPACES

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Abstract: Relative inductive dimensions and two new inductive dimensions for completely regular spaces are studied.

Key words: Relative dimension, relative realcompactness, Wallman realcompactification, zero-mapping, cozero-mapping.

AMS: 54F45 Ref. Ž.: 3.967.1

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0. Preliminaries. All given spaces are assumed to be completely regular. The collection of all zero-sets in X will be denoted by $Z(X)$. If $X \subseteq Y$, then $Z(X,Y)$ is the trace on X of the collection $Z(Y)$. Let $N(X)$ denote the family of all collections of the form $Z(X,Y)$ [1],[2]. Obviously each element of $N(X)$ is precisely a nest generated intersection ring in the sense of [3], a strong delta normal base in the sense of [4] and a zero-set structure in the sense of [5]. If $\mathcal{F} \in N(X)$, then $w(X,\mathcal{F})$ denotes the Wallman compactification and $v(X,\mathcal{F})$ - the Wallman realcompactification of X [3]. When there is no question as to the space X, we will simply write $w(\mathcal{F})$, $v(\mathcal{F})$. The space of real numbers is denoted by R.

The following definitions and propositions are given in [1],[2].

Definition 0.1. Let $X \subseteq Y$. We call a mapping $f: X \longrightarrow X'$ a $Z(X,Y)$-mapping if $f^{-1}(Z)$ is an element of the collection $Z(X,Y)$ for each zero-set Z of X'.

Definition 0.2. Let $X \subseteq Y$. We shall say that a space X is realcompact with respect to Y if $X = v(X,Z(X,Y))$.

Proposition 0.1. Let $\mathcal{F} \in N(X)$. $v(\mathcal{F})$ is the smallest space between X and $w(\mathcal{F})$, which is realcompact with respect to $w(\mathcal{F})$. In particular, X is realcompact with respect to $w(\mathcal{F})$ if and only if $X = v(\mathcal{F})$.

Proposition 0.2. Let $\mathcal{F} \in N(X)$ and $X \subseteq T \subseteq w(\mathcal{F})$. The following statements are equivalent.

(1) Every $Z(X,w(\mathcal{F}))$-mapping from X into any realcompact space Y has an extension to a $Z(T,w(\mathcal{F}))$-mapping from T into Y.

(2) Every $Z(X,w(\mathcal{F}))$-mapping from X into R has an extension to a $Z(T,w(\mathcal{F}))$-mapping from X into R.

(3) If a countable family of elements of the collection $\mathcal{F}$ has empty intersection, then their closures in T have empty intersection.

(4) For any countable family of elements F_n of the collection $\mathcal{F}$,

$$\left[\bigcap_{n=1}^{\infty} F_n\right]_T = \bigcap_{n=1}^{\infty} [F_n]_T .$$

(5) Every point of T is the limit of a unique, real, $\mathcal{F}$-ultrafilter on X.

(6) $X \subseteq T \subseteq v(\mathcal{F})$.

(7) $v(T,Z(T,w(\mathcal{F}))) = v(\mathcal{F})$.

Proposition 0.3. Let $\mathcal{F} \in N(X)$ and $F \in \mathcal{F}$. Then $[F]_{v(\mathcal{F})}$

is an element of the collection $Z(v(\mathcal{F}),w(\mathcal{F}))$ and $v(F,Z(F,w(\mathcal{F}))) = [F]_{v(\mathcal{F})}$.

Proposition 0.4. Let $\mathcal{F} \in N(X)$ and $F \in Z(v(\mathcal{F}),w(\mathcal{F}))$. Then $F = [F \cap X]_{v(\mathcal{F})}$.

1. Relative dimensions $I(X,Y)$ and $i(X,Y)$

Definition 1.1. Let $X \subseteq Y$. The relative large inductive dimension of X with respect to Y, denoted by $I(X,Y)$, is defined inductively as follows. $I(X,Y) = -1$ if and only if $X = \emptyset$. For a non-negative integer n, $I(X,Y) \leq n$ means that for each pair Z_1, Z_2 of disjoint elements of collection $Z(X,Y)$, there exist $Z \in Z(X,Y)$, $O_1, O_2 \in CZ(X,Y)$ with $X - Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ $(i = 1,2)$ and $I(Z,Y) \leq n - 1$. $I(X,Y) = n$ if $I(X,Y) \leq n$ and $I(X,Y) \not\leq n - 1$. $I(X,Y) = \infty$ means that there is no n for which $I(X,Y) \leq n$.

The relative small inductive dimension $i(X,Y)$ of X with respect to Y is defined by analogy with Definition 1.1.

These relative dimensions $I(X,Y)$, $i(X,Y)$ are topological invariants in the following sense: if f is a homeomorphism from Y onto any space Y' with $f(X) = X'$ $(X \subseteq Y)$, then $I(X,Y) = I(X',Y')$ and $i(X,Y) = i(X',Y')$. On the other hand, these relative dimensions are not topological invariants in the usual sense [1].

The following two lemmas are obvious.

Lemma 1.1. Let $X \subseteq T \subseteq Y$. If T is z-embedded [6] in Y, then $I(X,T) = I(X,Y)$ and $i(X,T) = i(X,Y)$.

Lemma 1.2. Let $X \subseteq Y$. If $Z \in Z(X,Y)$, then $I(Z,Y) \leq I(X,Y)$.

Lemma 1.3. Let $X \subseteq Y$. If a space X is the union of a sequence $\{D_i\}$ of disjoint sets such that the partial unions $\bigcup_{j \le i} D_j$ are elements of the collection $Z(X,Y)$, then $I(X,Y) \le \sup I(D_i,Y)$.

Proof. The proof of this lemma is similar to the proof of the Dowker's additive theorem for dimension Ind in completely normal spaces [7].

Lemma 1.4. Let $X \subseteq Y$. If $G \in CZ(X,Y)$, then $I(G,Y) \le$ $\le I(X,Y)$.

Proof. Let $I(X,Y) = k$. In case $k = -1$ the lemma holds clearly. We suppose that $k \le n$ and that the lemma holds for $k \le n - 1$.

Let $Z \in Z(G,Y)$ and $OZ \in CZ(G,Y)$ with $Z \subseteq OZ$. We may choose four sequences:

1. $\{Z_i\}_{i=1}^{\infty}$, $Z_i \in Z(X,Y)$, $i = 1,2,\dots$,

2. $\{O_i\}_{i=1}^{\infty}$, $O_i \in CZ(X,Y)$, $i = 1,2,\dots$,

3. $\{F_i\}_{i=1}^{\infty}$, $F_i \in Z(G,Y)$, $i = 1,2,\dots$,

4. $\{G_i\}_{i=1}^{\infty}$, $G_i \in CZ(G,Y)$, $i = 1,2,\dots$

with

$$Z_i \subseteq O_{i+1} \subseteq Z_{i+1} \subseteq G = \bigcup_{i=1}^{\infty} Z_i, \quad i = 1,2,\dots,$$

$$Z = \bigcap_{i=1}^{\infty} F_i \subseteq F_{i+1} \subseteq G_i \subseteq F_i \subseteq OZ, \quad i = 1,2,\dots .$$

By Lemma 1.2, $I(Z_{i+1},Y) \le n$ and hence there are $S_i \in Z(Z_{i+1},Y)$, $T_i \in CZ(Z_{i+1},Y)$, $i = 1,2,\dots$ with $Z \cap Z_i \subseteq T_i \subseteq S_i \subseteq G_i \cap O_{i+1}$ and $I(S_i - T_i,Y) \le n - 1$, $i = 1,2,\dots$. Evidently $T_i \in CZ(O_{i+1},Y)$ and hence $T_i \in CZ(G,Y)$, $i = 1,2,\dots$. Let $S = \bigcup_{i=1}^{\infty} S_i$, $T =$ $= \bigcup_{i=1}^{\infty} T_i$. We have $Z \subseteq T \subseteq S \subseteq OZ$, $T \in CZ(G,Y)$ and

$$S_i \subseteq \bigcap_{k=1}^{\infty}\{F_k \cup [\bigcup_{j<k} S_j]\} \subseteq \bigcap_{k=1}^{\infty} F_k \cup S, \ i = 1,2,\dots .$$

Hence $S = \bigcap_{k=1}^{\infty}\{F_k \cup [\bigcup_{j<k} S_j]\}$ and so S is an element of the collection $Z(G,Y)$.

Let $D_k = \bigcup_{i\le k}(S_i - T_i)$ and $D = \bigcup_{k=1}^{\infty} D_k$. Clearly, $D_{k+1} - D_k$ is an element of the collection $CZ(S_{k+1} - T_{k+1},Y)$ and by the induction hypothesis $I(D_{k+1} - D_k,Y) \le n - 1$. Then by Lemma 1.3, $I(D,Y) \le n - 1$. Finally, $S - T \in Z(D,Y)$ and so, by Lemma 1.2, $I(S - T,Y) \le n - 1$. Thus $I(G,Y) \le n$.

__Theorem 1.1__. (__The subspace theorem__.) If $M \subseteq N \subseteq X$, then $I(M,X) \le I(N,X)$.

__Proof__. Let $I(N,X) = k$. For $k = -1$ the result is trivial. We assume its validity for $k \le n - 1$ and suppose $k \le n$.

Let Z_1, Z_2 be disjoint elements of the collection $Z(M,X)$. There are elements F_1, F_2 of $Z(N,X)$ with $Z_i = F_i \cap M$ $(I = 1,2)$. Evidently, $N - (F_1 \cap F_2) = G \in CZ(N,X)$ and hence, by Lemma 1.4, $I(G,X) \le n$. There are $F \in Z(G,X)$, G_1, $G_2 \in CZ(G,X)$ with $G - F = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$, $F_i \cap G \subseteq G_i$ $(i = 1,2)$ and $I(F,X) \le n - 1$. Clearly, $G_i \in CZ(N,X)$ $(i = 1,2)$. Finally, let $F \cap M = Z$, $G_i \cap M = O_i$ $(i = 1,2)$. Then $M - Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \subseteq O_i$ $(i = 1,2)$, $Z \in Z(M,X)$, O_1, $O_2 \in CZ(M,X)$ and by the induction hypothesis $I(Z,X) \le I(F,X) \le n - 1$. Thus $I(M,X) \le n$.

__Theorem 1.2__. (__The countable sum theorem__.) Let $X \subseteq Y$. If $X = \bigcup_{i=1}^{\infty} Z_i$ with $Z_i \in Z(X,Y)$ and $I(Z_i,Y) \le n$ for all $i = 1,2,\dots$, then $I(X,Y) \le n$.

__Proof__. For $n = -1$ the result is trivial. We assume its validity for $n \le k - 1$ and suppose $n \le k$.

Let $D_j = \bigcup_{i\le j} Z_i$. Each D_j is an element of the collection

$Z(X,Y)$ and by the subspace theorem $I(D_{j+1} - D_j, Y) \leq$ $\leq I(Z_{j+1}, Y) \leq k$. Then by Lemma 1.3, $I(X,Y) \leq k$.

Theorem 1.3. If $M \subseteq N \subseteq X$, then $i(M,X) \leq i(N,X)$

Proof is obvious.

Theorem 1.4. If $X \subseteq Y \subseteq T$, then $i(X,Y) \leq i(X,T)$.

Proof. Let $i(X,T) = k$. For $k = -1$ the result is trivial. We assume its validity for $k \leq n - 1$ and suppose $k \leq n$.

Let $x \notin Z$ and $Z \in Z(X,Y)$. There is a zero-set F' in T such that $Z \subseteq F'$ and $x \notin F'$. Hence $F = F' \cap X$ is an element of the collection $Z(X,T)$ with $Z \subseteq F$ and $x \notin F$. There are O_1, $O_2 \in$ $\in CZ(X,T)$, $D \in Z(X,T)$ such that $X - D = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $x \in O_1$, $F \subseteq O_2$ and $i(D,T) \leq n - 1$. Clearly, $D \in Z(X,Y)$, O_1, $O_2 \in$ $CZ(X,Y)$ and by the induction hypothesis $i(D,Y) \leq i(D,T) \leq n - 1$. Thus $i(X,Y) \leq n$.

Theorem 1.5. If $A \cup B \subseteq Y$, then $I(A \cup B, Y) \leq I(A,Y) +$ $+ I(B,Y) + 1$.

Proof. Let $I(A,Y) = k_1$, $I(B,Y) = k_2$ and $A \cup B = X$. For $k_1 = k_2 = -1$ the result is trivial. Let $k_1 \leq n$, $k_2 \leq m$ and assume the theorem for the cases $k_1 \leq n$, $k_2 \leq m - 1$ and $k_1 \leq n - 1$, $k_2 \leq m$.

Let Z_1, Z_2 be disjoint elements of the collection $Z(X,Y)$. Choose $O_1, O_2 \in CZ(X,Y)$ and $F_1, F_2 \in Z(X,Y)$ with $Z_i \subseteq O_i \subseteq$ $\subseteq F_i$ $(i = 1,2)$ and $F_1 \cap F_2 = \emptyset$. Since $I(A,Y) \leq n$, there are G_1, $G_2 \in CZ(A,Y)$ and $D \in Z(A,Y)$ with $A - D = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$, $F_i \cap A \subseteq G_i$ $(i = 1,2)$ and $I(D,Y) \leq n - 1$. By Proposition 14 from [8], there are $V_1, V_2 \in CZ(X,Y)$ with $V_i \cap A = G_i$ $(i = 1,2)$ and $V_1 \cap V_2 = \emptyset$. Then $U_1 = (V_1 - F_2) \cup O_1$ and $U_2 = (V_2 - F_1) \cup O_2$

are disjoint elements of the collection $CZ(X,Y)$ with $Z_i \subseteq U_i$ $(i = 1,2)$ and $A - (U_1 \cup U_2) = D$. $I(A - (U_1 \cup U_2),Y) = I(D,Y) \leq$ $\leq n - 1$; by the subspace theorem, $I(B - (U_1 \cup U_2),Y) \leq m$. By the induction hypothesis $I(X - (U_1 \cup U_2),Y) \leq n + m$. Thus $I(X,Y) \quad n + m + 1$.

Theorem 1.6. If $A \cup B \subseteq Y$, then $i(A \cup B,Y) \leq i(A,Y) +$ $+ i(B,Y) + 1$.

Proof is similar to the proof of Theorem 1.5.

Theorem 1.7. If $\mathcal{F} \in N(X)$, then $I(X,w(\mathcal{F})) = I(v(\mathcal{F}),$ $w(\mathcal{F}))$.

Proof. The theorem follows from Proposition 0.3 and from the following lemma.

Lemma 1.5. Let $\mathcal{F} \in N(X)$. If two disjoint elements F_1, F_2 of the collection $\mathcal{F}$ can be separated by an element F of the collection $\mathcal{F}$, then $[F]_{v(\mathcal{F})}$ separates $[F_i]_{v(\mathcal{F})}$ $i =$ $= 1,2$.

Proof is trivial.

Theorem 1.8. If $X \subseteq Y$, then $i(X,Y) \leq I(X,Y)$.

Proof is trivial.

Definition 1.2. Let $X \subseteq Y$. The relative large inductive dimension modulo R, denoted by $R - I(X,Y)$, is defined inductively as follows. $R - I(X,Y) = -1$ if and only if X is realcompact with respect to Y. For a non-negative integer n, $R - I(X,Y) \leq n$ means that for each pair Z_1, Z_2 of disjoint elements of the collection $Z(X,Y)$, there are $Z \in Z(X,Y)$, O_1, $O_2 \in CZ(X,Y)$ with $X - Z = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $Z_i \quad O_i$ $(i = 1,$ $2)$ and $R - I(Z,Y) \leq n - 1$.

Theorem 1.9. If $\mathcal{F} \in N(X)$, then $R - I(X,w(\mathcal{F})) = I(v(\mathcal{F}) - X,w(\mathcal{F}))$.

Proof. a) $R - I(X,w(\mathcal{F})) \leq I(v(\mathcal{F}) - X,w(\mathcal{F}))$.

Let $I(v(\mathcal{F}) - X,w(\mathcal{F})) = k$. For $k = -1$ the result is trivial. We assume its validity for $k \leq n - 1$ and suppose $k \leq n$.

Let $Z_1, Z_2 \in \mathcal{F}$ and $Z_1 \cap Z_2 = \emptyset$. There are $V_1, V_2 \in C\mathcal{F}$, $T_1, T_2 \in \mathcal{F}$ with $Z_i \subseteq V_i \subseteq T_i$ $(i = 1,2)$ and $T_1 \cap T_2 = \emptyset$. By the propositions 0.2, 0.3, $[T_1]_{v(\mathcal{F})} \cap [T_2]_{v(\mathcal{F})} = \emptyset$ and $[T_i]_{v(\mathcal{F})} \in Z(v(\mathcal{F}),w(\mathcal{F}))$ $(i = 1,2)$. Clearly, $[T_i]_{v(\mathcal{F})} \cap (v(\mathcal{F}) - X) = F_i \in Z(v(\mathcal{F}) - X,w(\mathcal{F}))$ $(i = 1,2)$ and $F_1 \cap F_2 = \emptyset$. There are sets $F \in Z(v(\mathcal{F}) - X,w(\mathcal{F}))$, $G_1, G_2 \in CZ(v(\mathcal{F}) - X,w(\mathcal{F}))$ with $F_i \subseteq G_i$ $(i = 1,2)$, $G_1 \cap G_2 = \emptyset$, $(v(\mathcal{F}) - X) - F = G_1 \cup G_2$ and $I(F,w(\mathcal{F})) \leq n - 1$. By Proposition 14 from [8], there are $G_i' \in CZ(v(\mathcal{F}),w(\mathcal{F}))$ with $G_1' \cap G_2' = \emptyset$ and $G_i' \cap (v(\mathcal{F}) - X) = G_i$ $(i = 1,2)$. Let $U_1 = G_1' - [T_2]_{v(\mathcal{F})}$ and $U_2 = G_2' - [T_1]_{v(\mathcal{F})}$. Clearly, $U_1 \cap U_2 = \emptyset$, $U_1 \cap T_2 = \emptyset$, $U_2 \cap T_1 = \emptyset$, $U_i \cap (v(\mathcal{F}) - X) = G_i$ $(i = 1,2)$ and $U_i \in CZ(v(\mathcal{F}),w(\mathcal{F}))$ $(i = 1,2)$. Let $H_i = U_i \cup O_{v(\mathcal{F})}(V_i)$, where $O_{v(\mathcal{F})}(V_i) = v(\mathcal{F}) - [X - V_i]_{v(\mathcal{F})}$ $(i = 1,2)$. Clearly, $H_i \in CZ(v(\mathcal{F}),w(\mathcal{F}))$ and $H_i \cap (v(\mathcal{F}) - X) = G_i$ $(i = 1,2)$. Evidently, $Z_i \subseteq V_i \subseteq O_{v(\mathcal{F})}(V_i) \subseteq H_i$ $(i = 1,2)$ and $H_1 \cap H_2 = \emptyset$. Let $D' = v(\mathcal{F}) - (H_1 \cup H_2)$. We have $D' \in Z(v(\mathcal{F}),w(\mathcal{F}))$, $D' \cap (v(\mathcal{F}) - X) = F$ and hence by Proposition 0.4, $[D' \cap X]_{v(\mathcal{F})} = D'$ and $D' = D \cup F$, where $D = D' \cap X$. By Proposition 0.3, $[D]_{v(\mathcal{F})} = v(D,Z(D,w(\mathcal{F})))$ and hence $F = v(D,Z(D,w(\mathcal{F}))) - D$. Clearly, $D \in \mathcal{F}$, $H_i \cap X \in C\mathcal{F}$, $Z_i \subseteq H_i \cap X$ $(i = 1,2)$, $(H_1 \cap X) \cap (H_2 \cap X) = \emptyset$, $(H_1 \cap X) \cup (H_2 \cap X) = X - D$ and by the induction hypothesis, $R - I(D,w(\mathcal{F})) \leq$

$\leq I(F,w(\mathcal{F})) \leq n - 1$. Thus $R - I(X,w(\mathcal{F})) \leq n$.

b) $I(v(\mathcal{F}) - X,w(\mathcal{F})) \leq R - I(X,w(\mathcal{F}))$.

Let $R - I(X,w(\mathcal{F})) = k$. For $k = -1$ the result is trivial. We assume its validity for $k \leq n - 1$ and suppose $k \leq n$.

Let $Z_1,Z_2 \in Z(v(\mathcal{F}) - X,w(\mathcal{F}))$ and $Z_1 \cap Z_2 = \emptyset$. There are $Z_i' \in Z(v(\mathcal{F}),w(\mathcal{F}))$ with $Z_i' \cap (v(\mathcal{F}) - X) = Z_i$ $(i = 1,2)$. Let $Z = Z_1' \cap Z_2'$. Clearly, $Z \in \mathcal{F}$, $X - Z \in C\mathcal{F}$, $X - Z$ is dense in $v(\mathcal{F}) - Z$. It should be observed that each $Z(X - Z,w(\mathcal{F}))$-mapping from $X - Z$ into R has an extension to a $Z(v(\mathcal{F}) - Z,w(\mathcal{F}))$-mapping from $v(\mathcal{F}) - Z$ into R. This shows that by Proposition 0.2, $v(X - Z,Z(X - Z,w(\mathcal{F}))) = v(v(\mathcal{F}) - Z,Z(v(\mathcal{F}) - Z,w(\mathcal{F})))$. $v(\mathcal{F}) - Z$ is realcompact with respect to $w(\mathcal{F})$ and hence $v(\mathcal{F}) - Z = v(X - Z,Z(X - Z,w(\mathcal{F})))$.

Evidently,

(1) $v(X - Z,Z(X - Z,w(\mathcal{F}))) - (X - Z) = v(\mathcal{F}) - X$.

Clearly, $Z_i' \cap (X - Z) = F_i \in Z(X - Z,w(\mathcal{F}))$ $(i = 1,2)$ and $F_1 \cap F_2 = \emptyset$. There are $F \in Z(X - Z,w(\mathcal{F}))$, $O_1,O_2 \in CZ(X - Z,w(\mathcal{F}))$ with $(X - Z) - F = O_1 \cup O_2$, $O_1 \cap O_2 = \emptyset$, $F_i \subseteq O_i$ $(i = 1,2)$ and $R - I(F,w(\mathcal{F})) \leq R - I(X - Z,w(\mathcal{F})) - 1$. $X - Z \in C\mathcal{F}$ and hence, as in Lemma 1.4, $R - I(X - Z,w(\mathcal{F})) \leq R - I(X,w(\mathcal{F}))$. Finally, we have $R - I(F,w(\mathcal{F})) \leq n - 1$. By Lemma 1.5, $[F]_{v(\mathcal{F})-Z}$ separates $[F_1]_{v(\mathcal{F})-Z}$ and $[F_2]_{v(\mathcal{F})-Z}$. Then $D = [F]_{v(\mathcal{F})-Z} \cap \cap (v(\mathcal{F}) - X)$ separates Z_1 and Z_2. Finally, as it is shown in the part a) of this proof, $D = v(F,Z(F,w(\mathcal{F}))) - F$ and by the induction hypothesis, $I(D,w(\mathcal{F})) \leq R - I(F,w(\mathcal{F})) \leq n - 1$. Thus by (1), $I(v(\mathcal{F}) - X,w(\mathcal{F})) \leq n$.

<u>Remark 1</u>. It should be observed that the dimension

$R - I(X,Y)$ satisfies conditions which are similar to the countable sum theorem (theorem 1.2) and Lemma 1.4 respectively. On the other hand, $R - I(X,Y)$ is not monotone in general.

2. Inductive dimensions $\mathrm{Ind}_0 X$ and $\mathrm{ind}_0 X$

Definition 2.1. $\mathrm{Ind}_0 X = I(X,X)$, $\mathrm{ind}_0 X = i(X,X)$ and $R - \mathrm{Ind}_0 X = R - I(X,X)$.

Theorem 2.1. Ind_0, ind_0 and $R - \mathrm{Ind}_0$ are topological invariants.

Proof is trivial.

Theorem 2.2. $\mathrm{ind}_0 X \leq \mathrm{Ind}_0 X$.

Proof follows from the theorem 1.8.

Theorem 2.3. $\mathrm{ind}_0 X = \inf\{i(X,Y), X \subseteq Y\}$.

Proof follows from the theorem 1.4.

Theorem 2.4. If $X \subseteq Y$, then $\mathrm{ind}_0 X \leq \mathrm{ind}_0 Y$.

Proof. By Theorem 1.4, $\mathrm{ind}_0 X \leq i(X,Y)$; by Theorem 1.3, $i(X,Y) \leq i(Y,Y) = \mathrm{ind}_0 Y$. Thus $\mathrm{ind}_0 X \leq \mathrm{ind}_0 Y$.

The similar results (Theorems 2.3 and 2.4) are not true for the dimension Ind_0.

Theorem 2.5. If $X \subseteq Y$, then $I(X,Y) \leq \mathrm{Ind}_0 Y$. In particular, if X is z-embedded in Y, then $\mathrm{Ind}_0 X \leq \mathrm{Ind}_0 Y$.

Proof follows from the theorem 1.1 and Lemma 1.1.

Corollary 1. If G is a cozero-set in X, then $\mathrm{Ind}_0 G \leq \mathrm{Ind}_0 X$.

Theorem 2.6. If X is the countable union of zero-set subsets $\{Z_i\}_{i=1}^{\infty}$ with $I(Z_i,X) \leq n$ for all $i = 1,2,\dots$, then

$\mathrm{Ind}_0 X \leq n$. In particular, if each Z_i is z-embedded in X and $\mathrm{Ind}_0 Z_i \leq n$, then $\mathrm{Ind}_0 X \leq n$.

Proof follows from the countable sum theorem and Lemma 1.1.

Theorem 2.7. $\mathrm{Ind}_0 X = \mathrm{Ind}_0 vX$, where vX is the Hewitt realcompactification of X.

Proof follows from Theorem 1.7 and Lemma 1.1.

The following corollary gives a positive answer on the question 2 from [9] for pseudocompact spaces.

Corollary 2 [10]. If X is pseudocompact space, then $\mathrm{Ind}_0 X = \mathrm{Ind}_0 \beta X$ (βX is the Stone-Čech compactification of X).

Theorem 2.8. If the Hewitt realcompactification vX of X is Lindelöf, then $\mathrm{ind}_0 vX = \mathrm{Ind}_0 vX$.

Proof is similar to the Smirnov's theorem: $\mathrm{ind}\,\beta X = \mathrm{Ind}\,\beta X$ for perfectly normal X [11].

Corollary 3. If X is Lindelöf, then $\mathrm{ind}_0 X = \mathrm{Ind}_0 X$.

Theorem 2.9. $R - \mathrm{Ind}_0 X = I(vX - X, vX)$.

Proof follows from Theorem 1.9 and Lemma 1.1.

Corollary 4. If $vX - X$ is z-embedded in vX, then $\mathrm{Ind}_0(vX - X) = R - \mathrm{Ind}_0 X$.

Corollary 5. If X is a pseudocompact space satisfying the bicompact axiom of countability [12], then $\mathrm{ind}_0(\beta X - X) = R - \mathrm{Ind}_0 X = \mathrm{Ind}_0(\beta X - X)$.

Theorem 2.10. If $X = A \cup B$, then $\mathrm{Ind}_0 X \leq I(A,X) + I(B,X) + 1$ and $\mathrm{ind}_0 X \leq i(A,X) + i(B,X) + 1$.

Proof follows from Theorems 1.5 and 1.6.

It is shown in [13] that for each non-negative integer n there exists a completely regular space X^n with $X^n = X_1^n \cup \cup X_2^n$, X_1^n and X_2^n are the zero-sets of X^n, $\dim X_i^n = 0$ (i = = 1,2) and $\dim X^n = n$ (dimension dim is defined as in [14]). This example shows that "Urysohn Inequality" - $\mathrm{Ind}_0(A \cup B) \leq \leq \mathrm{Ind}_0 A + \mathrm{Ind}_0 B + 1$ does not hold in general (indeed, for an arbitrary completely regular space X we have: dim X $\mathrm{Ind}_0 X$ and "dim X = 0 if and only if $\mathrm{Ind}_0 X = 0$").

The following theorem gives a positive answer on the question 3 from [9] for pseudocompact spaces.

Theorem 2.11. For each pseudocompact space X with $\omega X = \tau$ and $\mathrm{Ind}_0 X \leq n$, there exists a compactification bX of X with $\omega bX = \tau$ and $\mathrm{Ind}_0 bX \leq n$.

Proof follows from Corollary 2 and from the following

Theorem [15]. If f is a continuous mapping from a bicompact X into a bicompact Y, then there exists a bicompact Z, continuous mappings $g: X \longrightarrow Z$ and $h: Z \longrightarrow Y$ such that $f = hg$, $\mathrm{Ind}_0 Z \leq \mathrm{Ind}_0 X$, $\omega Z \leq \omega Y$.

Definition 2.2. We call a mapping $f: X \longrightarrow Y$ a zero-mapping if f(Z) is a zero-set of the space Y for each zero-set Z of the space X.

The following theorem generalizes the well-known Hurewitz Theorem [16].

Theorem 2.12. Let f be a continuous zero-mapping of a space X onto a space Y such that the inverse image $f^{-1}(y)$ consists of at most k + 1 points for each point y of Y.

Then we have $\mathrm{Ind}_0 Y \leq \mathrm{Ind}_0 X + k$.

Proof is such as in [17].

Finally, we have the following generalization of the Alexandroff's theorem [18].

Theorem 2.13. Let f be a continuous cozero-, zero-mapping of a bicompact X onto a bicompact Y such that the inverse image $f^{-1}(y)$ consists of at most countable points for each point y of Y. Then we have $\mathrm{Ind}_0 X = \mathrm{Ind}_0 Y$.

Proof is such as in [19] (notion of a cozero-mapping is defined as in the definition 2.2).

Remark 2. It should be observed that the dimensions Ind_0 and ind_0 are equal to the dimensions Ind and ind respectively in the class of perfectly normal spaces.

References

[1] A.CH. CHIGOGIDZE: Relative dimensions for completely regular spaces, Bull. Acad. Sci. Georgian SSR 85(1977), 45-48.

[2] A.CH. CHIGOGIDZE: On the Wallman realcompactifications and dimensions of increments of completely regular spaces, Bull. Acad. Sci. Georgian SSR 87(1977) (to appear).

[3] A.K. STEINER and E.F. STEINER: Nest generated intersection rings in Tychonoff spaces, Trans. Amer. Math. Soc. 148(1970), 589-601.

[4] R.A. ALO and H.L. SHAPIRO and M. WEIR: Realcompactness and Wallman realcompactification, Portugal. Math. 34(1975), 33-43.

[5] H. GORDON: Rings of functions determined by zero-sets, Pacific J. Math. 36(1971), 133-157.

[6] A.W. HAGER: On inverse-closed subalgebras of C(X), Proc. London Math. Soc. 19(1969), 233-257.

[7] C.H. DOWKER: Inductive dimensions of completely normal spaces, Quart. J. Math. 4(1953), 267-281.

[8] R.N. ORMOTSADZE and A.CH. CHIGOGIDZE: Inductive dimensions for zero-set spaces, Bull. Acad. Sci. Georgian SSR 81(1976), 301-304.

[9] A.V. IVANOV: On the dimension of incompletely normal spaces, Vestnik Mosk. Univ. 4(1976), 21-27.

[10] A.CH. CHIGOGIDZE: On the pseudocompact spaces, Bull. Acad. Sci. Georgian SSR 86(1977), 25-27.

[11] IU.M. SMIRNOV: Some relations in the dimension theory, Matem. Sbor. 29(1951), 157-172.

[12] IU.M. SMIRNOV: On the dimension of increments of bicompact extensions of proximity spaces and topological spaces, Matem. Sbor. 69(1966), 141-160.

[13] J. TERESAWA: N ∪ R and their dimensions, Notices Amer. Math. Soc. 24(1977), A-262.

[14] R. ENGELKING: Outline of General Topology, Amsterdam, 1968.

[15] A.G. NEMETS and B.A. PASYNKOV: On two general approaches to the factorization theorems in the dimension theory, Dokl.AN SSSR 233(1977), 788-791.

[16] W. HUREWICH and H. WALLMAN: Dimension Theory, Princeton, 1941.

[17] K. MORITA: On closed mappings and dimensions, Proc. Japan Acad. 32(1956), 161-165.

[18] P.S. ALEXANDROFF: On the countably-order open mappings, Dokl. AN SSSR 4(1936), 283-286.

[19] B.A. PASYNKOV: On the open mappings, Dokl. AN SSSR 175 (1967), 292-295.

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(Oblatum 7.6. 1977)