Karel Čuda

The nonabsolute boundedness model of the theory of semisets

Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 4, 763--769

Persistent URL: http://dml.cz/dmlcz/105819

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

18,4 (1977)

THE NONABSOLUTE BOUNDEDNESS MODEL OF THE THEORY OF SEMISETS

Karel ČUDA, Praha

Abstract: A common description of minimal (in a sense) models of the theory of semisets is given. One of these models is considered in detail. We prove that in this model there is no semiset bijection between two natural numbers the ratio of which is a nonstandard natural number. In this model we can have a semiset bijection between any two nonstandard natural numbers with the ratio standard.

Key words: Absolute natural number, semiset, F-definition.

AMS: Primary 02K10, 02K05 Ref. Ž.: 2.641, 2.666
Secondary 10N15, 26A98

Introduction. The paper is the first of three papers concerning "minimal" models of the theory of semisets in itself. These models have a common description given by the metatheorem 1.01 below. The models are useful for investigating nonstandardness in the theory of semisets; the author believes that the models can be useful also for mathematicians interested in nonstandard analysis and nonstandard model of arithmetic. These are advised to look at the semisets as external objects and the sets as internal ones.

Preliminaries: TSS is the weak theory of semisets (see

[H]). Thus, it is a theory with three sorts of variables x, X^o, X subordinated in this order. The theory is the Gödel-Bernays set theory with respect to the sorts x, X^o and G.B. theory of classes with respect to the sorts x, X.

We put: $Dsg(X) \equiv (\exists X^o)(X = X^o)$ (" X is a designated class")

$M(X) \equiv (\exists x)(X = x)$ (" X is a set")

$Sm(X) \equiv (\exists x)(X \subseteq X)$ (" X is a semiset")

Semisets are usually designated by the lower case greek letters.

$Abs(X) \equiv (\forall \sigma \subseteq X)M(\sigma)$ (" X is absolute")

$(FN) \equiv (\exists X)(\forall n \in \omega)(n \in X \equiv Abs(n))$

There is a class of all absolute (that is, finitely large or standard) natural numbers.

$FN = \{n \in \omega ; Abs(n)\}$

We use the notation FN rather than An (see [C 3]) in accordance with the notation used in the alternative set theory (see [V] or [S])

$DEP(X,Y) \equiv (\exists R^o)(X = (R^o)"Y)$ $\quad$ $dep(X,Y) \equiv (\exists r)(X = r"Y)$

$DEP_d(X,Y) \equiv (\exists F^o, Fnc(F^o))(X = ((F^o)^{-1})"Y)$ $\quad$ $dep_d(X,Y) \equiv (\exists f, Fnc(f))(X = (f^{-1})"Y)$

These relations are often used in boolean models and properties of them are wellknown (see [VH]).

$BD(A,Y) \equiv (\exists Z \subseteq A)DEP(Y,Z)$ $\quad$ $bd(a,\sigma) \equiv (\exists \varphi \subseteq a)dep(\sigma,\varphi)$

"Y is bounded by A" $\quad$ " σ is bounded by a"

§ 1. Common characterization of "minimal" models

I.01 Metatheorem: Let $\varphi(\sigma)$ be a formula (it can have some parameters) such that in TSS + Γ (where Γ is a suitable system of axioms) the following can be proved:

1) $(\forall \sigma_1, \varphi(\sigma_1))(\forall \sigma_2, \varphi(\sigma_2))(\exists \rho, \varphi(\rho))\mathrm{dep}(\sigma_1 \times \sigma_2, \rho)$

2) $(\forall \sigma, \varphi(\sigma))(\forall a, a \supseteq \sigma)(\exists \rho, \varphi(\rho))\mathrm{dep}(P(a - \sigma), \rho)$

(That is: The "system" of semisets "described" by φ is "closed" under the cartesian product and the powerclass of the complement.) Then the F-definition (see [VH] 1262)

$\epsilon^* = \epsilon$

$\mathrm{Cls}^*(X) \equiv (\exists \sigma, \varphi(\sigma))\mathrm{DEP}(X, \sigma) \vee X = 0$

$\mathrm{Dsg}^*(X) \equiv \mathrm{Dsg}(X)$

describes a model of TSS + $(\forall X)((\exists a)\mathrm{BD}(a,X) \vee X = 0)$ in TSS + Γ . This model is a submodel of the identical model and the designated classes (and hence also the sets) are absolute.

Demonstration: To demonstrate the fact that the considered definition describes a model, it suffices to prove that new classes are closed under Gödelian operations. For the operations $\mathrm{Cnv}_2(X)$, $\mathrm{Cnv}_3(X)$, $E(X)$, $D(X)$ the fact we need is a consequence of the transitivity of DEP. To prove this for the operation $X \wedge Y$ let us note the following facts: $X \wedge Y = X \cap (V \times Y)$, $\mathrm{DEP}(X \cap Y, X \times Y)$, $\mathrm{DEP}(V \times Y, Y)$, $\mathrm{DEP}(X,S)$ & & $\mathrm{DEP}(Y,T) \Longrightarrow \mathrm{DEP}(X \times Y, S \times T)$, $\mathrm{dep}(\sigma, \rho) \equiv \mathrm{DEP}(\sigma, \rho)$ and the property 1) of the formula φ . To prove the required fact for the operation $X - Y$ note in addition $X - Y = X \cap \cap (V - Y)$, $\mathrm{DEP}(X, \sigma)$ & $a \supseteq \sigma$ & $X \neq V \Longrightarrow \mathrm{DEP}_d(V - X, P(a - \sigma))$ (cf [VH] 4107).

In order to prove the validity of $(\forall X)(\exists a)\ \mathrm{BD}(X,a)$ in the model it suffices to prove that for any class X of the model there is a semiset σ of the model such that

$DEP(X,\sigma)$ (absoluteness of DEP and sets). But the semisets having the property φ are in the model.

§ 2. The nonabsolute boundedness model

2.01 Definitions: 1) $bd\neg Abs(\sigma) \equiv (\forall n, \neg Abs(n)) bd(b,\sigma)$.

2) $BD\neg Abs(X) \equiv (\exists \sigma)(bd\neg Abs(\sigma) \& DEP(X,\sigma)) \vee X = 0$.

Fact: $BD\neg Abs(X) \Longrightarrow (\forall n, \neg Abs(n)) BD(n,X) \vee X = 0$.

(The converse implication also holds. The reader will easily prove this after reading the paper.)

2.02 Metatheorem: The F-definition from 1.01 with the specification $\varphi = bd\neg Abs(\sigma)$ gives an essentially faithful (see [VH] 1232) model of TSS + $(\forall X) BD\neg Abs(X)$ in TSS.

Remarks: 1) We will prove $(\forall X) BD\neg Abs(X) \Longrightarrow (FN)$.

2) Recall that the faithfulness of a model means that in the model the same as in the modeled theory can be proved.

3) The formula $(\forall X)(\exists a) BD(a,X)$ is an easy consequence of $(\forall X) BD\neg Abs(X)$.

Before demonstrating 2.02 let us note some facts.

2.03 Metalemma: The formula $bd(a,\sigma)$ is normal (see [VH] 1123) in TSS.

Demonstration: $bd(a,\sigma) \equiv (\exists r)(\forall x \in \sigma)(\exists y \in a)$
$(x \in r''\{y\} \& r''\{y\} \subseteq \sigma)$

2.04 Corollary: $(\exists \sigma) bd\neg Abs(\sigma) \& \neg M(\sigma) \Longrightarrow (FN)$.

Proof: $FN = \{n; \neg\ bd(n,\sigma)\}$.

2.05 Corollary: $(\forall X) BD\neg Abs(X) \Longrightarrow (FN)$.

Demonstration of 2.02: Let us verify the properties 1), 2) from 1.01. The property 1) is a consequence of the following two facts: a) For any nonabsolute natural number α there is a nonabsolute natural number β such that $\beta^2 \leq \alpha$ ($\mathrm{Abs}(n) \Longrightarrow \mathrm{Abs}(n^2)$ see [C 3]).

b) $\mathrm{bd}(\beta, \sigma) \& \mathrm{bd}(\beta, \varphi) \Longrightarrow \mathrm{bd}(\beta^2, \sigma \times \varphi)$.

In order to verify the property 2) note that:

a) $\mathrm{Abs}(a) \Longrightarrow \mathrm{Abs}(P(a))$ (see [C 3]).

b) $a \supset \varphi \& b \supset \sigma \& \mathrm{dep}(\varphi, \sigma) \Longrightarrow \mathrm{dep}_d(P(a - \varphi), P(b - \sigma))$ (as $\mathrm{dep}(P(a) - P(a - \varphi), \varphi)$ (see [VH] 4105) $\Longrightarrow \mathrm{dep}(P(a) - P(a - \varphi), \sigma) \Longrightarrow \mathrm{dep}_d(P(a) - P(a - \varphi), P(b) - P(b - \sigma))$ (see [VH] 4107) $\Longrightarrow \mathrm{dep}_d(P(a - \varphi), P(b - \sigma))$)). Using a), b) we easily prove the property 2) in 1.01.

Now, let us prove the validity of $(\forall X)\mathrm{BD} \neg \mathrm{Abs}(X)$ in the model. Let us note that any number nonabsolute in the sense of model is nonabsolute in the theory. (In the model there are fewer semisets than in the theory.) Now, let us prove another fact: If $\mathrm{bd}(a, \sigma)$ holds then there is a semiset $\varphi \subseteq a$ such that $\mathrm{dep}(\varphi, P(\sigma)) \& \mathrm{dep}(\sigma, \varphi)$. Let $b \supset \sigma = r''\bar{\varphi} \& \bar{\varphi} \subseteq a$. Define the relation $\bar{r} \subseteq a \times P(b)$ as follows: $\bar{r}''\{c\} = \{x; x \in a \& r''\{x\} = c\}$. Put $\varphi = \bar{r}''P(\sigma)$. Let σ be a semiset in the model and let $\mathrm{bd}(a, \sigma)$. $P(\sigma)$ is in the model and hence $\mathrm{bd}(a, \sigma)$ holds in the model. The validity $(\forall X)\mathrm{BD} \neg \mathrm{Abs}(X)$ in the model is an easy consequence of the absoluteness of designated classes and of the fact that semisets with the property $\mathrm{bd} \neg \mathrm{Abs}(\sigma)$ are in the model. (We can see that if there is a nondesignated class in the model, the absolute numbers in the theory and in the model coincide.)

To demonstrate that the model is essentially faithful it suffices to prove that the model is the identical model of TSS + $(\forall X\ BD\neg Abs(X)$ in itself (see [VH] 1236). Remember that $(\forall X)BD\neg Abs(X) \Longrightarrow (FN)$ (2.05), note that $BD\neg Abs(FN)$. Consequently FN is absolute. $bd\neg Abs(\sigma)$ is a normal formula with exactly one semiset parameter FN (2.03), using the absoluteness of FN and sets we obtain the absoluteness of this formula. The assertion is now an easy consequence of the absoluteness of the designated classes and the fact that any semiset σ having the property $bd\neg Abs(\sigma)$ is a member of the model.

Now we give some assertions valid in the model. We prove these assertions in the theory modeled.

2.06 Theorem: $(\forall \sigma)bd(n,\sigma) \Longrightarrow (\forall x,y)Appr(x,y,n+1)$. (The definition of $Appr(x,y,z)$ see [VH] 5304 .)
In words: Any semiset function is a part of a set "tube" with the diameter less than n + 1.

Proof: Let σ be a semiset function. Let $\sigma = r''\varphi$, where $\varphi \subseteq n$. We can suppose that $D(r) = n \,\&\, (\forall k \in n)Fnc(r''\{k\})$. We have $\sigma \subseteq r''n$.

Remark: Using $(\forall X)BD(n,X)$ we can generalize the given assertion for class functions.

2.07 Theorem TSS + $(\forall n, \neg Abs(n))(\forall x,y)Appr(x,y,n)$: Let k be a natural number and n be nonabsolute. There is no class bijection between k and n.k.

Proof: Let $\sigma : k \longleftrightarrow n.k$. Let r be an approximating relation with the diameter less than n. We have

$$n.k = card(r''k) \leq card(r) \leq (n-1).k$$

which is a contradiction.

Remarks: 1) It was the proof of the consistence of 2.07 with nonstandardness which led P. Vopěnka to the first description of the considered model.

2) For any nonabsolute natural number n and absolute k we can have a semiset bijection between n and n.k in the considered model.

R e f e r e n c e s

[C3] K. ČUDA: Contribution to the theory of semisets III, Zeitschr. f. math. Logik und Grundlagen d. Math. 19(1973), 399-406.

[H] P. HÁJEK: Why semisets?, Comment. Math. Univ. Carolinae 14(1973), 397-420.

[S] A. SOCHOR: The alternative set theory, Proceedings of the second colloquium in set theory and hierarchy theory, Springer-Verlag.

[V] P. VOPĚNKA: Mathematics in the alternative set theory, to appear.

[VH] P. VOPĚNKA, P. HÁJEK: The theory of semisets, Academia, Prague 1972 and North-Holland Publishing Company Amsterdam.

Matematický ústav
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 29.9. 1977)