Miroslav Hušek
Topological spaces without $\kappa$-accessible diagonal

Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 4, 777--788

Persistent URL: http://dml.cz/dmlcz/105821

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1977

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
TOPOLOGICAL SPACES WITHOUT \( \aleph \)-ACCESSIBLE DIAGONAL

M. HUŠEK, Praha

Abstract: Spaces which may replace in factorization situations spaces with \( G_{\aleph} \)-diagonal are investigated. Problems in special cases are connected with \( \mathbb{B} \mathbb{N} \) and metrizability of compact spaces.

Key words: \( \aleph \)-accessible diagonal, factorization of maps on products, cardinal functions, metrizability.

AMS: 54F99, 54D30, 54E35 Ref. 2. 3. 961

The following definition was motivated by results concerning factorization of maps on products of spaces. Some basic facts and ideas may be found in [Hu_1].

Definition. We shall say that a topological space \( X \) has a (weakly) \( \aleph \)-accessible diagonal if there is a net \( \{ a_\xi | \xi < \aleph \} \) in \( X \times X - \Delta_X \) (weakly) converging to diagonal \( \Delta_X \).

The fact that \( X \) has not (weakly) \( \aleph \)-accessible diagonal is denoted by \( \aleph \in \Delta_X \) (or \( \aleph \in \overline{\Delta_X} \), resp.). Thus \( \aleph \in \Delta_X \) (or \( \aleph \in \overline{\Delta_X} \)) iff for any net \( \{ a_\xi | \xi < \aleph \} \) in \( X \times X - \Delta_X \) there is a cofinal set \( C \) in \( \aleph \) and a neighborhood \( U \) of \( \Delta_X \) in \( X \times X \) such that \( U \cap \{ a_\xi | \xi < \aleph \} \subseteq C \) is empty.

Since \( \aleph \in \Delta_X \) iff \( \text{cof} \aleph \in \Delta_X \) (the same for \( \overline{\Delta_X} \)),
it suffices to restrict a consideration to regular cardinals; for then, \( \omega \in \Delta X \) (\( \omega \in \overline{\Delta} X \)) iff for any \( M \subset X \times X \) with \( |M - \Delta_X| = \omega \) there is a neighborhood \( U \) of \( \Delta_X \) in \( X \times X \) with \( |M - U| = \omega \) (\( |M - \overline{U}| = \omega \), resp.).

The spaces without \( \omega \)-accessible diagonal (\( \omega \)-regular) generalize spaces \( X \) with \( \psi(\Delta_X, X \times X) < \omega \) (e.g., if \( X \) has a \( G_{\delta} \)-diagonal (or \( \overline{G}_{\delta} \)-diagonal), then \( \omega_1 \in \Delta X \) (\( \omega_1 \in \overline{\Delta} X \), resp.)).

E. van Douwen after discussion with the author about spaces \( X \) with \( \omega_1 \in \Delta X \) (Amsterdam 1975) called them "spaces with small diagonal". In the meantime, the author used in several lectures (also in [Hu2]) the terms "D-spaces, \( D_1 \)-spaces" for \( X \) with \( \omega \in \Delta X \), \( \omega_1 \in \Delta X \). In this time we are convinced that the term "spaces without \( \omega \)-accessible diagonal" is more justified.

After stating general results we shall restrict our consideration to the cases \( \omega = \omega \), \( \omega = \omega_1 \). In the sequel, a topological space always means a Hausdorff one, \( \omega \) denotes a regular infinite cardinal. We shall omit \( X \times X \) in \( \psi(\Delta_X, X \times X) \) and similar expressions.

1. Observations. (1) \( \psi(\Delta_X) < \omega \rightarrow \omega \in \Delta X \).
(2) \( \chi(\Delta_X) = \psi(\Delta_X) \rightarrow \omega \notin \Delta X \).
(3) \( \omega \in \Delta X \rightarrow \omega \notin \{ \alpha \mid \alpha = \psi(x) = \chi(x) \text{ for some } x \in X \} \).
(4) If \( X \) is compact then (2) and (3) means: \( \omega = \omega X \) or \( \omega \in \{ \psi(x) \mid x \in X \} \rightarrow \omega \notin \Delta X \).
(5) \( \overline{\Delta} X \subset \Delta X \).
The converse implications in (1) - (4) do not hold even for compact $X$ (for $\omega = \omega_0$ one may take $\beta \mathbb{N}$ in (1) and $2^{\omega_1}$ in (4)). In (5) the equality holds if $\Delta_X$ has a base of closed neighborhoods, which is true e.g. if all neighborhoods of $\Delta_X$ form a uniformity.

**Proposition 1.** The class of spaces without $\omega$-accessible diagonal is hereditary, $\lambda$-productive for any $\lambda < \omega$, and the property is preserved by taking larger topologies.

Clearly, $2^{\omega}$ has $\omega$-accessible diagonal and hence, we cannot put $\lambda = \omega$ in Proposition 1.

In the sequel we shall use the term "$\omega$-compactness" in the following meaning: any subset of cardinality at least $\omega$ has an accumulation point (i.e., any closed discrete subspace is of cardinality less than $\omega$). Any $\omega$-compact spaces is pseudo-$\omega$-compact in the sense of Isbell. The concept corresponding to pseudo-$\omega, \lambda$-compactness is $(\omega, \lambda)$-compactness here: any subset $A$ of cardinality at least $\omega$ has a $\lambda$-accumulation point $x$ (i.e., for any neighborhood $U$ of $x$, $|U \cap A| \leq \lambda$).

**Theorem 1.** If $X$ is a $\omega$-compact space, then it has not $\omega$-accessible diagonal iff any continuous $f: \prod_{i} X_i \rightarrow X$, $\prod_{i} X_i$ $\omega$-compact, depends on less than $\omega$ coordinates.

**Proof.** Suppose first that $\omega \in \Delta X$, $\prod_{i} X_i$ is $\omega$-compact, $f: \prod_{i} X_i \rightarrow X$ is continuous not depending on less than $\omega$ coordinates. Then $\{i \in I | f(x_i) \neq f(y_i) \text{ for some } x_i, y_i \in \prod_{i} X_i \text{ with } pr_{I-(i)}x_i = pr_{I-(i)}y_i \} \leq \omega$ (denote this subset of $I$ by $J$). There are a neighborhood $U$ of $\Delta_X$ and
Let $x$ be an accumulation point of \( \{x_i \mid i \in J'\} \) in $\biguplus_{i} X_i$, $V$ its canonical neighborhood such that $f(V) \cap f(V') \subset U$. There is an $i \in J'$ such that $x_i \in V$, $pr_i(V) = X_i$; consequently, $y_i \in V$, and $\langle fx_i, fy_i \rangle \in U$, which is a contradiction.

Suppose now that $\mathcal{Y} \notin \Delta X$, i.e., there exists a set $A = \{ \langle x_\xi, y_\xi \rangle \mid \xi < \omega \}$ in $X \times X - \Delta_X$ converging to $\Delta_X$. Put $X_{-1}$ to be the set $A \cup \Delta_X$ with the following topology: $A$ is an open discrete subspace of $X_{-1}$, neighborhoods of points from $\Delta_X$ are traces on $X_{-1}$ of their neighborhoods in $X \times X$. It is almost self-evident that $X_{-1}$ is $\omega$-compact. Now, $X_{-1} \times 2^{\omega \omega}$ is $\omega$-compact and the following map $f: X_{-1} \times 2^{\omega \omega} \rightarrow X$ is continuous and does not depend on less than $\omega$ coordinates:

$$f(\langle x_\xi, y_\xi \rangle, \{k_\xi \mid \xi < \omega\}) = \begin{cases} x_\xi & \text{if } k_\xi = 0, \\ y_\xi & \text{if } k_\xi = 1, \end{cases}$$

$$f(\langle x, x \rangle, \{k_\xi \mid \xi < \omega\}) = x.$$ 

In the first part of the proof, $\omega$-compactness of $X$ was not used, but we must realize that by investigating factorizations of $f$ we are interested only in $f(\biguplus_{i} X_i)$. Hence, the restriction on $X$ in Theorem 1 is no restriction if we want $\biguplus_{i} X_i$ to be $\omega$-compact.

The most general condition which may be posed on $\biguplus_{i} X_i$ in the above factorization theorems is pseudo-$\omega$-compactness ([NU] for uncountable $\omega$, [Hu$_1$] for $\omega = \omega$). In that case the situation is more complicated, and we know only the following result:
Theorem 2. Each of the following conditions implies the next one:

(1) $X$ has not weakly $\omega$-accessible diagonal (i.e., $\omega \in \Delta X$).

(2) Any continuous $f: \prod_{i} X_i \to X$, $\prod_{i} X_i$ pseudo-$\omega$-compact, depends on less than $\omega$ coordinates.

(3) $X$ has not $\omega$-accessible diagonal (i.e., $\omega \in \Delta X$).

Proof is similar to the preceding one. (See [Hu1] for details of $(1) \implies (2)$. To prove $(2) \implies (3)$, one may take in the proof of Theorem 1 the subspace $A \cup (\Delta \cap \Delta X)$ of $X_{-1}$ as the new $X_{-1}$; if $A$ converges to $\Delta X$, then this new $X_{-1}$ is pseudo-$\omega$-compact. The remaining procedure is the same.

The implication $(2) \implies (1)$ is not true in general. Clearly, if $\Delta X = \Delta X$, then all the three conditions are equivalent. We do not know whether $(3) \implies (2)$ (in fact, we do not know any example of a pseudo-$\omega$-compact space $X$ with $\omega \in \Delta X - \bar{F}X$).

In the second part of the proof of Theorem 1 we used the index set of cardinality $\omega$; in such a case we may prove more:

Theorem 3. If $X$ has not $\omega$-accessible diagonal, then any continuous map $f: Y \to X$, where $Y$ is a $(\omega, \omega)$-compact subspace of a $\omega$-fold product $\prod_{\eta} X_{\eta}$, depends on less than $\omega$ coordinates.

Proof. Suppose that an $f$ from our theorem does not depend on less than $\omega$ coordinates. Then we can find points $x_{\xi}, y_{\xi}$ in $Y$ for $\xi < \omega$ with $pr_{\eta} x_{\xi} = pr_{\eta} y_{\xi}$ for all $\eta \in \xi$. 

- 781 -
and $fx \neq fy$. Thus for a cofinal $C$ in $\mathcal{A}$ and a neighborhood $U$ of $\Delta_X$ we have $U \cap \{<fx, fy> | \xi \in C \} = \emptyset$. Let $x \in X$ be a $\mathcal{A}$-accumulation point of $\{x_\xi | \xi \in C \}$, $V$ its canonical neighborhood such that $f(V \cap X) \times f(V \cap X) \subset U$. There is a $\xi \in C$ such that $x_\xi \in V$ and $pr_2 V = X_\eta$ provided $\eta \geq \xi$; hence, $y_\xi \in V$ -- a contradiction.

From the results of the second section we shall see that Theorem 3 is not valid for more than $\mathcal{A}$-fold products; if $2^\omega = \omega_1$, $X = \beta \omega$, then $X$ may be embedded into $[0,1]^{\omega_1}$ and the identity $1_X$ does not depend on countably many coordinates.

It is not difficult to show that if $X$ is compact, then $\mathcal{A} \in \Delta_X$ iff $X \times X - \Delta_X$ is $(\mathcal{A}, \mathcal{A})$-compact.

At the end of the first part we shall remark that if $X$ is a scattered compact space, then $\Delta X = |X|^+, \rightarrow [X]$. Indeed, if $A$ is an infinite subset of $X$, $x_0$ is a complete accumulation point of $A$ with the least order, $U$ is a closed neighborhood of $x_0$ with $U \cap x | \text{order of } x \leq \text{order of } x_0 = (x_0)$, then $U \cap A$ converges as a well-ordered net of type $|A|$ to $x_0$.

2. In this part we shall be interested in the case $\mathcal{A} = \omega$. The earlier results have now simpler formulations, mainly for compact spaces:

Theorem 4. The following are equivalent for a compact space $X$:

1. $X$ has not $\omega$-accessible diagonal.
2. $X \times X - \Delta_X$ is countably compact.
(3) Any continuous map \( f: \prod_{i} X_i \to X \), \( \prod_{i} X_i \) pseudomapping (or compact), depends on finitely many coordinates.

(4) Any continuous map \( f: Y \to X \), where \( Y \) is a countably compact subspace of a countable product, depends on finitely many coordinates.

If \( X \) has not \( \omega \)-accessible diagonal, then it has no convergent nontrivial sequence and, hence, nondiscrete metrizable spaces, infinite dyadic compact spaces, infinite Eberlein compact spaces, infinite scattered compact spaces, infinite supercompact spaces [DM] have \( \omega \)-accessible diagonal. The space \( \beta N \) with doubled \( N \) has \( \omega \)-accessible diagonal and no convergent nontrivial sequence.

It seems that for compact spaces, only finite ones have not \( \omega \)-accessible diagonal. The next result shows that there are many nontrivial compact spaces without \( \omega \)-accessible diagonal. The result appeared in [Hu1].

**Theorem 5.** If any countable discrete set in a completely regular space \( X \) is \( C^* \)-embedded in \( X \), then \( X \) has not weakly \( \omega \)-accessible diagonal.

**Proof.** Suppose \( \{<x_n^*, y_n>\}^\omega \subset X \times X - \Delta X \). If one of the points \( x_n, y_n \) appears infinitely many times, e.g. all \( x_n \) equal to \( x_0 \), then for suitable neighborhoods \( U, V \) of \( x_0 \), \( \overline{V} \cap \overline{U} \) misses infinitely many of \( y_n \)'s, the set \( X \times (X - \overline{V}) \cup (U \times U) \) is a neighborhood of \( \Delta X \) the closure of which misses infinitely many of \( <x_n, y_n> \)'s. In the other case we can choose a subsequence \( \{<u_n, v_n>\} \) of \( \{<x_n, y_n>\} \) such that the sets \( \{u_n\} = A \), \( \{v_n\} = B \) are disjoint and discrete in \( X \); moreover, we may suppose that \( A \cup B \) is dis-
crete (if $B \subseteq A$, then there is infinite $A_1 \subseteq A$ with $A_1 \cap B = \emptyset$ because $A$ is $C^\ast$-embedded). Then $\overline{A}^{\beta X} \cap \overline{B}^{\beta X} = \emptyset$ and, consequently, $\overline{A}^{\beta X} \times \overline{B}^{\beta X}$ is separated from $\Delta_{\beta X}$ in $\beta X \times \beta X$.

It does not suffice to suppose that any countable subset of $X$ contains a $C^\ast$-embedded infinite subset: put $X$ to be $\beta N$ with doubled $N$.

There are compact spaces without $\omega$-accessible diagonal containing a set having no $C^\ast$-embedded (in $X$) infinite subset (e.g. the compactification of $N$ from the Example 5.22 [W] obtained as a quotient of $\beta N$ along an idempotent permutation).

**Corollary.**

(1) If $D$ is a discrete space, then no subspace of $\beta D$ has weakly $\omega$-accessible diagonal.

(2) No extremally disconnected space has $\omega$-accessible diagonal.

In (2) we may put basically disconnected or moreover F-spaces instead of extremally disconnected spaces. The class of spaces without $\omega$-accessible diagonal is bigger than that of F-spaces because the former class is finitely productive (or use the example just before Corollary). We do not know whether any compact space without $\omega$-accessible diagonal can be embedded into a countable (hence finite) product of F-spaces.

**Theorem 6.** If $X$ is an infinite compact space without $\omega$-accessible diagonal, then $|X| \geq 2^{\omega_1}$.

Proof follows from a theorem of Čech and Pospíšil because $X$ contains an infinite compact subspace $Y$ without isolated points (since $X$ is not scattered) and $\chi(x,Y) \geq \omega_1$.
for any $x \in X$.

As follows from results in [M§1], the last Theorem can be improved under MA: If $X$ is an infinite compact space without $\omega$-accessible diagonal, then $|X| \geq 2^{2^\omega}$.

It is an open problem whether there exists a compact space of cardinality $2^\omega$ without $\omega$-accessible diagonal. We are not sure that one can use the Fedorčuk's construction of a compact space of cardinality $2^\omega$ and without convergent nontrivial sequences.

At the end of this part we want to stress the fact that if a compact space without $\omega$-accessible diagonal is embedded into a countable product, then it can be embedded into a finite subproduct. This result is related to a recent deeper but more special result by V. Malyhin (unpublished): If $\beta N$ is embedded into a countable product then it can be embedded into one member of the product.

3. The case $\omega = \omega_1$ has in a sense "opposite" problems than the countable case. We do not know whether there are nonmetrizable compact spaces without $\omega_1$-accessible diagonal (or pseudo-$\omega_1$-compact spaces without both $G_\delta$-diagonal and $\omega_1$-accessible diagonal). This is important to know because up to now we do not know whether the factorization result in Theorem 1 is a generalization of the known result (the range has $G_\delta$-diagonal).

E. van Douwen proved that any compact linearly ordered space without $\omega_1$-accessible diagonal is metrizable, and D. Lutzer improved this for Lindelöf instead of com-
Theorem 7. (CH) A compact space is metrizable iff it has not \( \omega_1 \)-accessible diagonal and one of the following conditions holds:

(a) \( dX = \omega \)
(b) \( tX = \omega \)
(c) \( wX \leq 2^\omega \) or \( |X| \leq 2^\omega \)
(d) \( |C(X)| \leq 2^\omega \)

Proof. Suppose \( X \) is a compact space without \( \omega_1 \)-accessible diagonal. Then (c) clearly implies metrizability of \( X \). Since (a) \( \Rightarrow \) (d), it will suffice to prove that (d) implies metrizability and (b) \( \Rightarrow \) (a). Under (d), \( X \hookrightarrow [0,1]^\omega \), thus by Theorem 3, \( X \hookrightarrow [0,1]^\omega \). Suppose now that \( tX = \omega \). If \( X \) is not separable, then there is a set \( A = \{ x_\xi \mid \xi < \omega_1 \} \) such that \( x_\eta \notin \{ x_\xi \mid \xi < \eta \} \) for all \( \eta < \omega_1 \). Since \( tX = \omega \), we have \( X = \bigcup_{\xi \in \omega_1} \{ x_\xi \mid \xi < \eta \} \) and, by preceding considerations, all \( \{ x_\xi \mid \xi < \eta \} \) are metrizable. Hence \( |A| \leq 2^\omega \) and \( A \) is metrizable, hence hereditarily separable — a contradiction.

Questions. (1) Is there a compact nonmetrizable space without \( \omega_1 \)-accessible diagonal? Under CH, this question is equivalent to the following one: Is there a nonmetrizable compactification \( X \) of the discrete space \( \omega_1 \) such that \( X \) has not \( \omega_1 \)-accessible diagonal? (Any separable subspace of \( X \) must be metrizable.)

If one can prove that any compact space without \( \omega_1 \)-accessible diagonal is first countable, then it is metrizable without using CH (\( X \times X \) has not \( \omega_1 \)-accessible diagonal).
nal, the quotient of $X \times X$ along $\Delta_x$ has not $\omega_1$-accessible diagonal).

(2) Has $\beta N$ always a convergent net of type $\omega_1$? Equivalently: Is there always an ultrafilter on $N$ that can be expressed as a union of strictly increasing family of $\omega_1$-filters? (Our conjecture: it is consistent with ZFC that there is no such ultrafilter on $N$ (perhaps under $\text{MA} + \neg \text{CH}$?).

At the end we want to remark that I. Juhász has recently come to a similar problem: Is there a compact space $X$ with $\chi(X) > \omega$ and with no convergent nontrivial net of type $\omega_1$? This question is related to the problem of omitting $\omega_2$ by compact spaces (see $[J_2]$).

References


Matematický ústav
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 5.10.1977)