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WHAT TO EMBED INTO A CARTESIAN CLOSED TOPOLOGICAL CATEGORY
(Preliminary communication)

Jíří ADÁMEK, Václav KOUBEK, Praha

Abstract: Herrlich and Nel [4] ask whether every topological category is a finitely productive subcategory of a cartesian closed one. We answer this in the negative and we characterize all such subcategories by a "smallness" condition.

Key words: Initially complete, cartesian closed, topological.

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I. Characterization. All categories are here considered to be concrete with finite concrete products; all subcategories to be full, finitely productive and concrete. The underlying set of an object $A$ is denoted by $|A|$; hom-sets in $X$ are denoted by $\mathcal{K}(A,B)$ ($\subseteq |B|^{|A|}$).

Categories, used by topologists, have a lot of common properties. Several authors have introduced axioms for these categories; the first was Hušek in [5]. We shall use Herrlich's notion of topological category [3]: this is a category which has

(i) projective generation [given objects $A_i$ and maps $f_i: X \rightarrow |A_i|$, $i \in I$, there exists an object $A$ on $X$...
such that for each map \( h: |B| \to X \) we have: \( h \in \mathcal{K}(B,A) \) iff \( f_i.h \in \mathcal{K}(B,A_i) \) for each \( i \); 

(ii) small fibres [for every set \( X \) all objects \( A \) with \( |A| = X \) form a small set]; 

(iii) constants [each \( \mathcal{K}(A,B) \) contains all constant maps from \( |A| \) to \( |B| \)].

While Antoine proves in [21] that every concrete category is a subcategory of a cartesian closed one, we are interested in subcategories of CCT (cartesian closed topological) categories. We find a necessary and sufficient condition for the existence of a CCT supercategory. By an important result of Herrlich and Nel [4] this is equivalent to the existence of a canonical (minimal) CCT supercategory, called CCT hull.

Let a category \( \mathcal{K} \) be given. A structured map into a set \( X \) is a pair \((f,V)\) consisting of an object \( V \) and a map \( f: |V| \to X \). Two such pairs \((f,V)\) and \((g,W)\) are equivalent if for every map \( h:X \to |R|, R \) an object, we have:

\[ h.f \in \mathcal{K}(V,R) \text{ iff } h.g \in \mathcal{K}(W,R). \]

They are productively equivalent if for each object \( U \) the structured maps \((f \times 1, V \times U)\) and \((g \times 1, W \times U)\) are equivalent. Then we write \((f,V) \approx (g,W)\).

**Definition.** A category is strictly small-fibred if
for every set $X$ there exists, up to $\approx$, only a set of structured maps onto $X$.

**Example.** Every small-fibred category with quotients which are finitely productive is strictly small-fibred.

**Theorem.** A topological category is isomorphic to a subcategory of a cartesian closed topological category iff it is strictly small-fibred.

**Counterexample.** The following category is topological but not strictly small-fibred.

**Objects:** pairs $(X,H)$ where $X$ is a set and $H$ is a set of pairs $(M,m)$ consisting of a subset $M \subseteq X$ and a power $m \leq \text{card } M$, subject to the condition:

$(\emptyset,0) \in H$ and $(\{x\},0), (\{x\},1) \in H$ for each $x \in X$.

**Morphisms** from $(X,H)$ to $(Y,K)$: maps $f: X \rightarrow Y$ such that $(M,m) \in H$ implies $(f(M)n) \in K$ where $n = \min (m, \text{card } f(M))$.

The proof of necessity in the above theorem is easy. Sufficiency is proved by the following construction.

**II. Construction.** Given a category $\mathcal{C}$ we define a new category $\mathcal{C}^*$.

**Objects** are pairs $(X,A)$ where $X$ is a set and $A$ is a class of structured maps into $X$ which is a union of a set $(1)$ of equivalence classes of the productive equivalence $\approx$ (i.e., $A = \overline{A}_0$ for a set $A_0 \subseteq A$, where $\overline{\text{barr}}$ denotes the closure with respect to $\approx$).

**Morphisms** are defined inductively, forming a class $\bigcup \mathcal{X}_i^*$ (the union ranging over all ordinals $i$).
$\mathcal{K}_0^k$ consists of maps of the form $f \cdot h : U \rightarrow (Y, B)$, $U \in \mathcal{K}$, where $h \in \mathcal{K}(U, V)$ and $(f, V) \in B$.

\[ \begin{array}{ccc}
U & \xrightarrow{f} & (Y, B) \\
\downarrow h & & \downarrow f \\
& V & \rightarrow (X, A_0) \xrightarrow{p} (Y, B) \end{array} \]

$\mathcal{K}^*_i$ is the least class, closed to composition, which contains maps $p : (X, A_0) \rightarrow (Y, B)$ such that $p \cdot f : V \rightarrow (Y, B)$ is in $\mathcal{K}^*_i$ for each $(f, V) \in A_0$.

$\mathcal{K}^*_\gamma = \bigcup_{i<\gamma} \mathcal{K}^*_i$ for each limit ordinal $\gamma$.

The category $\mathcal{K}^*$ has the following properties (of which only the first requires a somewhat technical proof).

1. $\mathcal{K}^*$ has finite products: $(X, A) \times (Y, B) = (X \times Y, A \times B)$ where $A \times B = \{ (f \times g, V \times W) \mid (f, V) \in A \text{ and } (g, W) \in B \}$.  

2. $\mathcal{K}$ is a dense subcategory of $\mathcal{K}^*$ (full, finitely productive), closed to projective generation.

3. $\mathcal{K}^*$ is cocomplete and for each object $(X, A)$ the endofunctor  
$(Y, B) \mapsto (Y, B) \times (X, A)$ preserves colimits.

4. If $\mathcal{K}$ is strictly small-fibred then $\mathcal{K}^*$ is cartesian closed and small-fibred and has projective generation.

5. If $\mathcal{K}$ is topological then $\mathcal{K}^*$ is CCT.

References

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