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SEMIGROUPS FOR WHICH EVERY TOTALLY IRREDUCIBLE S-SYSTEM IS INJECTIVE

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Abstract: We characterize those semigroups for which every totally irreducible S-system is injective. Also obtained are homological characterizations of semilattices of groups and commutative regular semigroups.

Key words: Totally irreducible, regular, injective, p-injective.

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O. Introduction. In recent years there have been many investigations into homological properties of semigroups and S-systems. Many of the questions asked are analogous to those from ring and R-module theory. For example, Fountain [3], extending the work of Feller and Gantos [2], characterized those monoids S for which every S-system is injective. This corresponded to the well-known theorem that a ring R is semisimple Artinian if and only if every R-module is injective. The fact that another equivalent condition, namely that every cyclic R-module is injective, does not carry over to semigroups was shown by Johnson and McMorris in [5].

The present note is concerned with characterizing those semigroups for which every totally irreducible S-system is injective. We obtain an analogous theorem to that of
Michler and Villamayor [7]. As a consequence we also obtain the analogue of a theorem of Kaplansky characterizing commutative regular rings. In addition we give a new homological characterization of semilattices of groups which can then be added to the list as given by Lajos [6].

In this paper, $S$ is a monoid with zero.

A unital right $S$-system $M_S$ with zero is a set $M$ with a multiplication $M \times S \rightarrow M$ given by $(m,s) \mapsto ms$ such that $m(s_1 s_2) = (ms_1)s_2$ and satisfying $m \cdot 1 = m$ for all $m \in M$ and having a distinguished element $\theta \in M$ satisfying $\theta s = \theta$ for all $s \in S$. We will denote this element, as well as the zero of $S$ by $0$.

An $S$-system $M_S$ is injective if for every $S$-monomorphism $f: A_S \rightarrow B_S$ and $S$-homomorphism $g: A_S \rightarrow M_S$ there is an $S$-homomorphism $h: B_S \rightarrow M_S$ satisfying $h \circ f = g$.

An $S$-subsystem $N_S$ of $M_S$ is essential in $M_S$ if every $S$-congruence on $M$ whose restriction to $N$ is the identity, is itself the identity on $M$. Note that if $N_S$ is essential in $M_S$ then $N_S \land K_S \neq 0$ for all non-zero $S$-subsystems $K_S$ of $M_S$.

Berthiaume [1] has shown that each $S$-system $M_S$ has a unique (up to isomorphism over $M_S$) essential extension $M_S$ called the injective hull of $M_S$.

For a ring $R$ with identity, Michler and Villamayor [7] have shown that the following statements are equivalent:

1. Every proper right ideal is an intersection of maximal right ideals;
2. Every simple right $R$-module is injective.

A right $S$-system $M_S$ is totally irreducible if the
only right S-congruences are the universal congruence \( \omega_M \) and the identity congruence \( i_M \), and \( M \neq 0 \). Note that if \( M_S \) is totally irreducible then \( M_S \) has no proper S-subsystems. Also, since \( S \) has an identity, every congruence is modular so Theorem 6.2 of Hoehnke [4] reads that \( M_S \) is totally irreducible if and only if \( M_S \cong S/\mu \) where \( \mu \) is a maximal right congruence on \( S \).

Finally, if \( f : A_S \rightarrow B_S \) is an S-homomorphism, the kernel congruence, \( \ker f \), on \( A_S \) is given by
\[
\ker f = \{ (x, y) \mid f(x) = f(y) \}
\]
Clearly \( \ker f \) is an S-congruence on \( A_S \).

1. Monoids whose totally irreducible S-systems are injective

Given a congruence \( \varphi \) on \( S \), let \( I(\varphi) \) denote the 0-class of \( \varphi \):
\[
I(\varphi) = \{ x \in S \mid (x, 0) \notin \varphi \}
\]

1.1. Theorem: The following conditions are equivalent:

(1) For every proper congruence \( \varphi \) on \( S \), \( I(\varphi) = \bigcap_{C \in C} I(\varphi) \) where \( C \) is the family of all maximal right congruences on \( S \) which contain \( \varphi \).

(2) Every totally irreducible S-system is injective.

Proof: If \( l = 0 \), there is nothing to prove, so we shall assume that \( l \neq 0 \).

(1) \( \Rightarrow \) (2): Let \( M \) be a totally irreducible S-system, let \( 0 \neq x \in \hat{M} \) where \( \hat{M} \) is the injective hull of \( M \), and define \( \lambda : S \rightarrow \hat{M} \) by \( \lambda(s) = xs \). Then \( \ker \lambda \) is a proper right con-
gruence on $S$. Let $\{\varphi_\alpha \mid \alpha \in \Lambda\}$ be the family of maximal right congruences on $S$ which contain $\ker \lambda$. Let $M_\infty = S/\varphi_\infty$ and define $\mu : xS \rightarrow \prod_\alpha \varphi_\alpha M_\infty$ by $\mu(xs) = ([s]_\infty)$ where $[s]_\infty$ is the equivalence class of $s$ in $M_\infty$. Consider $p_\infty \circ \mu$ where $p_\infty : \prod_\alpha \varphi_\alpha M_\infty \rightarrow M_\infty$ is the projection mapping. Suppose that $p_\infty \circ \mu$ is not one-to-one for all $\alpha \in \Lambda$. Since $M$ is essential in $\hat{M}$ and is totally irreducible, $(0) \neq M = M \cap xS \subseteq xS$ and so $\ker (p_\infty \circ \mu) | M = \omega_M$ for all $\alpha \in \Lambda$. Thus if $xs \in M \cap xS$, $\mu(xs) = 0$ and so $s \in I(\varphi_\infty)$ for all $\alpha \in \Lambda$. Thus $s \in \bigcap_{\alpha \in \Lambda} I(\varphi_\alpha) = I(\ker (\lambda))$ and so $\lambda(s) = xs = 0$. Consequently $M = xS \cap M = (0)$, a contradiction. Thus there exists an $\alpha \in \Lambda$ such that $p_\alpha \circ \mu$ is one-to-one. Then $xS \cong xS \cap M$ and so $xS$ is totally irreducible. Hence $M = xS \cap M = xS$ and $x \in M$, therefore $M = \hat{M}$.

(2) $\Rightarrow$ (1): Let $\varphi$ be a proper right congruence on $S$ and let $\mathcal{C}$ be the family of all maximal right congruences on $S$ which contain $\varphi$. Let $x \in S \setminus I(\varphi)$, and $\varphi_0$ be a right congruence on $S$ maximal with respect to $\varphi \leq \varphi_0$ and $(x, 0) \notin \varphi_0$. Let $J \subseteq S$ be the right ideal of $S$ which is a union of classes such that $J/\varphi_0 = [x] S$ where $[x]$ is the $\varphi_0$ class of $x$. Then $J/\varphi_0$ is totally irreducible for if $\sigma$ is a congruence on $J$, $\sigma \supseteq \varphi_0$, then $\gamma = \sigma \cup \varphi_0 | S \setminus J$ is a congruence on $S$ properly containing $\varphi_0$. Thus $(x, 0) \notin \gamma$ and so $\sigma = \omega_{J/\varphi_0}$. Thus $J/\varphi_0$ is totally irreducible; and so $J/\varphi_0$ is injective. Then we have the diagram
where $\alpha$ is the inclusion mapping. Let $\varnothing = \{ (a, b) \in S \times S \mid \phi[a] = \phi[b] \}$, then $\varnothing \supseteq \varnothing_0$ is a congruence on $S$. If $\varnothing \neq \varnothing_0$, then $(x, 0) \in \varnothing$ and $\{ x \} = \phi[\{ x \}] = \{ 0 \}$ and $(x, 0) \in \varnothing_0$, a contradiction. Thus $\varnothing = \varnothing_0$ and $\ker \phi = I(S/\varnothing_0)$. Therefore, $S/\varnothing_0 \cong J/\varnothing_0$ and $S/\varnothing_0$ is totally irreducible, and $\varnothing_0$ is a maximal congruence on $S$ containing $\varnothing$. Hence $x \notin I(\varnothing_0)$ so $x \notin \bigwedge \epsilon_\varnothing \bigwedge I(\varnothing_\alpha)$. Thus $x \notin \bigwedge \epsilon_\varnothing \bigwedge I(\varnothing_\alpha) = I(\varnothing).

Remark: Using methods similar to those above, we can prove that if each proper congruence $\varnothing$ on $S$ is the intersection of the family of all maximal congruences containing $\varnothing$, then every totally irreducible $S$-system is injective. However the converse is false as seen by considering a group with zero.

The next theorem is the semigroup analogue of Kaplansky's result which states that a commutative ring $R$ with identity is regular if and only if every simple $R$-module is injective.

1.2. Theorem: Let $S$ be a commutative monoid. $S$ is regular if and only if each totally irreducible $S$-system is injective.

Proof: Suppose each totally irreducible $S$-system is injective. Let $a \in S \setminus a^2 S$ and $\alpha = (a^2 S \setminus a^2 S) \cup I_\varnothing$. Let $\varnothing$ be
a maximal congruence containing $\mathcal{C}$. If $(a,0) \not\in \mathcal{C}$, then $[a]S = S/\mathcal{C}$ since $S/\mathcal{C}$ is totally irreducible. Thus $[a]S = [a]s = [as]$ for some $s \in S$, and so $(1,as) \in \mathcal{C}$. Since $\mathcal{C}$ is a congruence $(a,a^2s) \in \mathcal{C}$, but then $(a,0) \in \mathcal{C}$ since $(a^2s,0) \in \mathcal{C}$, a contradiction. Hence, $(a,0) \in \mathcal{C}$ for every maximal congruence $\mathcal{C} \supseteq \mathcal{C}$ so

$$a \in \mathcal{C} \bigcap I(\mathcal{C}) = I(\mathcal{C}) = a^2S$$

where $C = \{ \mathcal{C} \supseteq \mathcal{C} \mid \mathcal{C}$ is a maximal right congruence on $S \}$. Thus $ae a^2S$ for all $a \in S$ so $S$ is regular.

Conversely, let $M_S$ be totally irreducible. Then there is a maximal right congruence $\mathcal{C}$ on $S$ with $M \cong S/\mathcal{C}$. A theorem of Öehmke [9] says that $S/\mathcal{C}$ is either a group or the two element semilattice. Schein [11] defines an order $a \leq b$ on $M$ if $a \leq b$ where $E$ is the set of idempotents of $S$. Moreover, $B \subseteq M$ is compatible if for every $b \in B$ there is an $e_b \in E$ with $b e_b = b$ and $b e_c = c e_b$ for all $c \in B$. A face of $B \subseteq M$ is an element $a \in M$ with $a \leq b$ for all $b \in B$ and $as = at$ whenever $Bs = Bt$ for $s, t \in S$. Schein [11] proved that $M$ is injective if and only if every compatible subset of $M$ has a face. Clearly every group and the two element semilattice are injective by Schein's result and thus $M \cong S/\mathcal{C}$ is injective.

2. A generalization. In the theory of rings with identity, an $R$-module $M$ is injective if and only if each $R$-homomorphism from a right ideal of $R$ to $M$ has an extension to all of $R$. These two concepts do not coincide in the theory of semigroups as shown by Berthiaume [1].
**Definition:** An S-system $M_S$ is *weakly injective* if each S-homomorphism $f : A \rightarrow M$ from a right ideal of $S$ to $M$ has an extension $\hat{f} : S \rightarrow M$.

An S-system $M_S$ is *p-injective* if each S-homomorphism $f : aS \rightarrow S$ from a principal right ideal of $S$ to $M$ has an extension $\hat{f} : S \rightarrow M$.

Note that since $S$ has an identity 1, if $f(1) \neq m$, then $f(s) = ms$ and $\hat{f}$ is given by left multiplication by $m$. In this section we characterize monoids $S$ each of whose cyclic S-systems is p-injective and use this to generalize Theorem 1.2.

2.1. **Theorem** (Ming [8]): For a monoid $S$, the following are equivalent:

1. $S$ is regular.
2. Every S-system is p-injective.
3. Every cyclic S-system is p-injective.

The proof found in [8] carries over directly.

2.2. **Theorem:** $S$ is regular and $SaS aS$ for all $a \in S$ if and only if every totally irreducible S-system is p-injective and every right ideal is two-sided.

*Proof:* If $S$ is regular, then every S-system is p-injective by Theorem 2.1. Moreover, if $J$ is a right ideal of $S$ and $a \in J$, then $SaS aS \subseteq J$ and $J$ is two-sided.

Conversely, if every right ideal is two-sided, then $aS$ is a right ideal, $a \in aS$ and so $SaS aS$. To see that $S$ is regular, let $b \in S$. If $b$ is not regular, then $(1, b) \notin \Sigma = (bS \times bS) \cup I$ for otherwise $(1, b) \in \Sigma$ implies that $(1, 0) \in \Sigma$ and $\Sigma = \omega_S$. Thus $l = bs$ for some $s \in S$ and $b = bsb$. 
Likewise if $\lambda : S \rightarrow bS$ is given by $\lambda(s) = bs$, then $(1, b) \notin \ker \lambda$ for otherwise $(1, b) \in \ker \lambda$ implies $b = b^2$ and so $b$ is regular. Let $\varphi$ be a congruence maximal with respect to $\varphi \supseteq \alpha \cup \ker \lambda$ and $(1, b) \notin \varphi$. If $\varphi \subseteq \gamma$, $(1, b) \in \gamma$ but $(b, 0) \in \alpha \subseteq \varphi \subseteq \gamma$ so $\gamma = \omega_S$, thus $\varphi$ is a maximal right congruence, and so $S/\varphi$ is totally irreducible. Let $\psi : bS \rightarrow S/\varphi$ be defined by $\psi(bs) = [s]$, the equivalence class of $s$ determined by $\varphi$. Since $S/\varphi$ is $p$-injective, there is some $c \in S$ with $\psi(bt) = [c]bt$ for all $t \in S$. Thus $[c]b = \psi(b) = \psi(b \cdot 1) = [1]$ or $(1, cb) \in \varphi$. Now $cb \in Sb \subseteq bS$ so $(cb, 0) \in \alpha \subseteq \varphi$ and so $(1, 0) \in \varphi$. Then $\varphi \supseteq \omega_S$, a contradiction.

Remark: The conditions of Theorem 2.2 are equivalent to the fact that every $N$-class of $S$ is a right group (Petrich [10], p. 118).

2.3. Corollary: $S$ is a semilattice of groups if and only if every totally irreducible $S$-system is $p$-injective and every one sided ideal is two sided.

2.4. Corollary: Let $S$ be commutative, then $S$ is regular if and only if every totally irreducible $S$-system is injective.

In a future note, we plan to investigate those semigroups for which every cyclic $S$-system is injective.

References


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