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SEMIGROUPS FOR WHICH EVERY TOTALLY IRREDUCIBLE S-SYSTEM IS INJECTIVE

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Abstract: We characterize those semigroups for which every totally irreducible S-system is injective. Also obtained are homological characterizations of semilattices of groups and commutative regular semigroups.

Key words: Totally irreducible, regular, injective, p-injective.

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0. Introduction. In recent years there have been many investigations into homological properties of semigroups and S-systems. Many of the questions asked are analogous to those from ring and R-module theory. For example, Fountain [3], extending the work of Feller and Gantos [2], characterized those monoids S for which every S-system is injective. This corresponded to the well-known theorem that a ring R is semisimple Artinian if and only if every R-module is injective. The fact that another equivalent condition, namely that every cyclic R-module is injective, does not carry over to semigroups was shown by Johnson and McMorris in [5].

The present note is concerned with characterizing those semigroups for which every totally irreducible S-system is injective. We obtain an analogous theorem to that of

Michler and Villamayor [7]. As a consequence we also obtain the analogue of a theorem of Kaplansky characterizing commutative regular rings. In addition we give a new homological characterization of semilattices of groups which can then be added to the list as given by Lajos [6].

In this paper, S is a monoid with zero.

A unital right S-system M_S with zero is a set M with a multiplication $M \times S \longrightarrow M$ given by $(m,s) \longmapsto ms$ such that $m(s_1 s_2) = (ms_1)s_2$ and satisfying $m \cdot 1 = m$ for all $m \in M$ and having a distinguished element $\theta \in M$ satisfying $\theta s = \theta$ for all $s \in S$. We will denote this element, as well as the zero of S by 0.

An S-system M_S is injective if for every S-monomorphism $f: A_S \longrightarrow B_S$ and S-homomorphism $g: A_S \longrightarrow M_S$ there is an S-homomorphism $h: B_S \longrightarrow M_S$ satisfying $h \circ f = g$.

An S-subsystem N_S of M_S is essential in M_S if every S-congruence on M whose restriction to N is the identity, is itself the identity on M. Note that if N_S is essential in M_S then $N_S \cap K_S \neq 0$ for all non-zero S-subsystems K_S of M_S.

Berthiaume [1] has shown that each S-system M_S has a unique (up to isomorphism over M_S) essential extension $\hat{M}_S$ called the injective hull of M_S.

For a ring R with identity, Michler and Villamayor [7] have shown that the following statements are equivalent: (1) Every proper right ideal is an intersection of maximal right ideals; (2) Every simple right R-module is injective.

A right S-system M_S is totally irreducible if the only right S-congruences are the universal congruence ω_M and the identity congruence i_M, and $M_S \neq 0$. Note that if M_S is totally irreducible then M_S has no proper S-subsystems. Also, since S has an identity, every congruence is modular so Theorem 6.2 of Hoehnke [4] reads that M_S is totally irreducible if and only if $M \cong S/\mu$ where μ is a maximal right congruence on S.

Finally, if $f: A_S \longrightarrow B_S$ is an S-homomorphism, the <u>kernel congruence</u>, ker f, on A_S is given by $\ker f = \{(x,y) \mid f(x) = f(y)\}$. Clearly ker f is an S-congruence on A_S.

1. Monoids whose totally irreducible S-systems are injective

Given a congruence ρ on S, let $I(\rho)$ denote the 0-class of ρ :

$$I(\rho) = \{x \in S \mid (x,0) \in \rho\}$$

1.1. <u>Theorem</u>: The following conditions are equivalent:

(1) For every proper congruence ρ on S, $I(\rho) = \bigcap_{\sigma \in C} I(\sigma)$ where C is the family of all maximal right congruences on S which contain ρ .

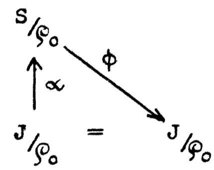
(2) Every totally irreducible S-system is injective.

Proof: If 1 = 0, there is nothing to prove, so we shall assume that $1 \neq 0$.

(1) $\Longrightarrow$ (2): Let M be a totally irreducible S-system, let $0 \neq x \in \hat{M}$ where $\hat{M}$ is the injective hull of M, and define $\lambda : S \longrightarrow \hat{M}$ by $\lambda(s) = xs$. Then $\ker \lambda$ is a proper right con-

gruence on S. Let $\{\sigma_\alpha \mid \alpha \in \Lambda\}$ be the family of maximal right congruences on S which contain ker λ. Let $M_\alpha = S/\sigma_\alpha$ and define $\mu : xS \longrightarrow \prod_{\alpha \in \Lambda} M_\alpha$ by $\mu(xs) = ([s]_\alpha)$ where $[s]_\alpha$ is the equivalence class of s in M_α. Consider $p_\alpha \circ \mu$ where $p_\alpha : \prod_{\alpha \in \Lambda} M_\alpha \longrightarrow M_\alpha$ is the projection mapping. Suppose that $p_\alpha \circ \mu$ is not one-to-one for all $\alpha \in \Lambda$. Since M is essential in $\hat{M}$ and is totally irreducible, $(0) \neq M = M \cap xS \subseteq xS$ and so ker $(p_\alpha \circ \mu) \mid_M = \omega_M$ for all $\alpha \in \Lambda$. Thus if $xs \in M \cap xS$, $\mu(xs) = 0$ and so $s \in I(\sigma_\alpha)$ for all $\alpha \in \Lambda$. Thus $s \in \bigcap_{\alpha \in \Lambda} I(\sigma_\alpha) = I(\ker(\lambda))$ and so $\lambda(s) = xs = 0$. Consequently $M = xS \cap M = (0)$, a contradiction. Thus there exists an $\alpha \in \Lambda$ such that $p_\alpha \circ \mu$ is one-to-one. Then $xS \cong M_\alpha$ and so xS is totally irreducible. Hence $M = xS \cap M = xS$ and $x \in M$, therefore $M = \hat{M}$.

(2) $\Longrightarrow$ (1): Let ρ be a proper right congruence on S and let C be the family of all maximal right congruences on S which contain ρ. Let $x \in S \setminus I(\rho)$, and ρ_0 be a right congruence on S maximal with respect to $\rho \subseteq \rho_0$ and $(x,0) \notin \rho_0$. Let $J \subseteq S$ be the right ideal of S which is a union of ρ_0 classes such that $J/\rho_0 = [x]S$ where $[x]$ is the ρ_0 class of x. Then J/ρ_0 is totally irreducible for if σ is a congruence on J, $\sigma \supsetneqq \rho_0$, then $\gamma = \sigma \cup \rho_0 \mid_{S-J}$ is a congruence on S properly containing ρ_0. Thus $(x,0) \in \gamma$ and so $\sigma = \omega_{J/\rho_0}$. Thus J/ρ_0 is totally irreducible; and so J/ρ_0 is injective. Then we have the diagram



where α is the inclusion mapping. Let $\sigma = \{(a,b) \in S \times S \mid \phi[a] = \phi[b]\}$, then $\sigma \supseteq \rho_0$ is a congruence on S. If $\sigma \neq \rho_0$, then $(x,0) \in \sigma$ and $[x] = \phi[x] = [0]$ and $(x,0) \in \rho_0$, a contradiction. Thus $\sigma = \rho_0$ and ker $\phi = i_{S/\rho_0}$. Therefore, $S/\rho_0 \cong J/\rho_0$ and S/ρ_0 is totally irreducible, and ρ_0 is a maximal congruence on S containing ρ. Hence $x \notin I(\rho_0)$ so $x \notin \bigcap_{\alpha \in \Lambda} I(\sigma_\alpha)$. Thus $\bigcap_{\alpha \in \Lambda} I(\sigma_\alpha) = I(\rho)$.

Remark: Using methods similar to those above, we can prove that if each proper congruence ρ on S is the intersection of the family of all maximal congruences containing ρ, then every totally irreducible S-system is injective. However the converse is false as seen by considering a group with zero.

The next theorem is the semigroup analogue of Kaplansky's result which states that a commutative ring R with identity is regular if and only if every simple R-module is injective.

1.2. <u>Theorem</u>: Let S be a commutative monoid. S is regular if and only if each totally irreducible S-system is injective.

Proof: Suppose each totally irreducible S-system is injective. Let $a \in S \setminus a^2S$ and $\alpha = (a^2S \times a^2S) \cup i_S$. Let ρ be a maximal congruence containing α . If $(a,0) \notin \rho$, then $[a]S = S/\rho$ since S/ρ is totally irreducible. Thus $[1] = [a]s = [as]$ for some $s \in S$, and so $(1,as) \in \rho$. Since ρ is a congruence $(a,a^2s) \in \rho$, but then $(a,0) \in \rho$ since $(a^2s,0) \in \alpha \subseteq \rho$, a contradiction. Hence, $(a,0) \in \rho$ for every maximal congruence $\rho \supseteq \alpha$ so

$$a \in \bigcap_{\rho \in C} I(\rho) = I(\alpha) = a^2S \text{ where}$$

$C = \{\rho \supseteq \alpha \mid \rho \text{ is a maximal right congruence on } S\}$. Thus $a \in a^2S$ for all $a \in S$ so S is regular.

Conversely, let M_S be totally irreducible. Then there is a maximal right congruence ρ on S with $M \cong S/\rho$. A theorem of Oehmke [9] says that S/ρ is either a group or the two element semilattice. Schein [11] defines an order $a \leq b$ on M if $a \in bE$ where E is the set of idempotents of S. Moreover, $B \subseteq M$ is <u>compatible</u> if for every $b \in B$ there is an $e_b \in E$ with $b\,e_b = b$ and $b\,e_c = c\,e_b$ for all $c \in B$. A <u>face</u> of $B \subseteq M$ is an element $a \in M$ with $a \geq b$ for all $b \in B$ and $as = at$ whenever $Bs = Bt$ for s, $t \in S$. Schein [11] proved that M is injective if and only if every compatible subset of M has a face. Clearly every group and the two element semilattice are injective by Schein's result and thus $M \cong S/\rho$ is injective.

2. <u>A generalization</u>. In the theory of rings with identity, an R-module M is injective if and only if each R-homomorphism from a right ideal of R to M has an extension to all of R. These two concepts do not coincide in the theory of semigroups as shown by Berthiaume [1].

Definition: An S-system M_S is weakly injective if each S-homomorphism $f:A \longrightarrow M$ from a right ideal of S to M has an extension $\hat{f}:S \longrightarrow M$.

An S-system M_S is p-injective if each S-homomorphism $f:aS \longrightarrow S$ from a principal right ideal of S to M has an extension $\hat{f}:S \longrightarrow M$.

Note that since S has an identity 1, if $\hat{f}(1) = m$, then $f(s) = ms$ and $\hat{f}$ is given by left multiplication by m. In this section we characterize monoids S each of whose cyclic S-systems is p-injective and use this to generalize Theorem 1.2.

2.1. Theorem (Ming [8]): For a monoid S, the following are equivalent:

(1) S is regular.

(2) Every S-system is p-injective.

(3) Every cyclic S-system is p-injective.

The proof found in [8] carries over directly.

2.2. Theorem: S is regular and $Sa \subseteq aS$ for all $a \in S$ if and only if every totally irreducible S-system is p-injective and every right ideal is two-sided.

Proof: If S is regular, then every S-system is p-injective by Theorem 2.1. Moreover, if J is a right ideal of S and $a \in J$, then $Sa \subseteq aS \subseteq J$ and J is two-sided.

Conversely, if every right ideal is two-sided, then aS is a right ideal, $a \in aS$ and so $Sa \subseteq aS$. To see that S is regular, let $b \in S$. If b is not regular, then $(1,b) \notin \alpha =$ $= (bS \times bS) \cup i_S$ for otherwise $(1,b) \in \alpha$ implies that $(1,0) \in$ $\in \alpha$ and $\alpha = \omega_S$. Thus $1 = bs$ for some $s \in S$ and $b = bsb$.

Likewise if $\lambda : S \longrightarrow bS$ is given by $\lambda(s) = bs$, then $(1,b) \notin \ker \lambda$ for otherwise $(1,b) \in \ker \lambda$ implies $b = b^2$ and so b is regular. Let ρ be a congruence maximal with respect to $\rho \supseteq \alpha \cup \ker \lambda$ and $(1,b) \notin \rho$. If $\rho \subsetneq \gamma$, $(1,b) \in \gamma$ but $(b,0) \in \alpha \subseteq \rho \subseteq \gamma$ so $\gamma = \omega_S$, thus ρ is a maximal right congruence, and so S/ρ is totally irreducible. Let $\psi : bS \to S/\rho$ be defined by $\psi(bs) = [s]$, the equivalence class of s determined by ρ. Since S/ρ is p-injective, there is some $c \in S$ with $\psi(bt) = [c]\,bt$ for all $t \in S$. Thus $[c]\,b = \psi(b) = \psi(b \cdot 1) = [1]$ or $(1,cb) \in \rho$. Now $cb \in Sb \subseteq bS$ so $(cb,0) \in \alpha \subseteq \rho$ and so $(1,0) \in \rho$. Then $\rho \supseteq \omega_S$, a contradiction.

Remark: The conditions of Theorem 2.2 are equivalent to the fact that every N-class of S is a right group (Petrich [10], p. 118).

2.3. <u>Corollary</u>: S is a semilattice of groups if and only if every totally irreducible S-system is p-injective and every one sided ideal is two sided.

2.4. <u>Corollary</u>: Let S be commutative, then S is regular if and only if every totally irreducible S-system is injective.

In a future note, we plan to investigate those semigroups for which every cyclic S-system is injective.

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