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THEOREMS ON MAPPINGS SATISFYING A RATIONAL INEQUALITY

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Abstract: Mappings S and T of a metric space X into itself satisfying inequalities are shown to be either identical constant mappings or to have a unique common fixed point.

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The following theorem was given in a paper by M.S. Khan [4]

Theorem 1. Let S and T be mappings of the complete metric space X into itself such that

$$d(Sx,Ty) \le c\,\frac{d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)}{d(x,Ty) + d(y,Sx)}$$

for all x, y in X, where $0 \le c < 1$. Then S and T have a unique common fixed point z.

It was later shown in [1] that the theorem was incorrect as stated and needed the extra condition that $d(x,Ty) + d(y,Sx) = 0$ implies that $d(Sx,Ty) = 0$ for the theorem to hold.

In the following we consider mappings S and T satisfying a similar inequality. First of all we have

Theorem 2. Let S and T be mappings of the metric space X into itself such that for all x, y in X, either

$$d(Sx,Ty) \leq \frac{cd(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$ and $0 \leq c \leq 1$, or

$$d(Sx,Ty) = 0$$

otherwise. Then S and T are identical constant mappings on X.

Proof: Let x be an arbitrary point in X. Then if $d(STx,Tx) \neq 0$, we have

$$d(STx,Tx) \leq \frac{cd(Tx,STx)d(x,Tx)}{d(Tx,STx) + d(x,Tx)} .$$

It follows that

$$d(STx,Tx) \leq (c - 1)d(x,Tx),$$

giving a contradiction, since $c \leq 1$. We must therefore have $STx = Tx$ for all x in X and so $ST = T$.

We can prove similarly that $TS = S$. Thus

$$d(Tx,STx) + d(Sy,TSy) = 0$$

for all x, y in X, which implies that $d(STx,TSy) = 0$ for all x, y in X. It follows that ST and TS are identical constant mappings and so S and T are identical constant mappings. This completes the proof of the theorem.

We now prove

Theorem 3. Let S and T be mappings of the complete metric space X into itself such that for all x, y in X, either

$$d(Sx,Ty) \leq \frac{cd(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$ and $1 < c < 2$, or

$$d(Sx,Ty) = 0$$

otherwise. Then each of S and T has a unique fixed point and these points coincide.

<u>Proof</u>: Let x be an arbitrary point in X and put

$$u_{2n} = d((ST)^n x, T(ST)^n x), \quad u_{2n+1} = d(T(ST)^n x, (ST)^{n+1} x)$$

for $n = 0,1,2,\ldots$.

Suppose first of all that $u_{2n} + u_{2n+1} = 0$ for some n. Then it follows immediately that $z = (ST)^n x$ is a common fixed point of S and T. Similarly $u_{2n-1} + u_{2n} = 0$ for some n implies that $z = T(ST)^{n-1}x$ is a common fixed point of S and T.

Now suppose that $u_n + u_{n+1} \neq 0$ for $n = 0,1,2,\ldots$. Then

$$u_n \leq \frac{cu_{n-1}u_n}{u_{n-1} + u_n}$$

and so

$$u_n \leq (c-1)u_{n-1} \leq (c-1)^2 u_{n-2} \leq (c-1)^n u_0.$$

Since $1 < c < 2$, it follows that the sequence

$$\{x, Tx, STx, \ldots, (ST)^n x, T(ST)^n x, \ldots\}$$

is a Cauchy sequence in the complete metric space X and so has a limit z in X.

If we now suppose that $Tz \neq z$ then

$$d((ST)^n x, Tz) \leq \frac{cu_{2n-1} d(z,Tz) + bd(T(ST)^{n-1}x, Tz) d(z,(ST)^n x)}{u_{2n-1} + d(z,Tz)}$$

and on letting n tend to infinity we have

$$d(z,Tz) \leq 0,$$

giving a contradiction. It follows that z is a fixed point of T.

We can prove similarly that z is also a fixed point of S and so z is a common fixed point of S and T.

Now suppose that T has a second fixed point z'. Then $d(z,Sz) + d(z',Tz') = 0$ and so

$$d(Sz,Tz') = 0 = d(z,z').$$

It follows that $z = z'$ and so T has a unique fixed point z. Similarly, we can prove that z is a unique fixed point of S. This completes the proof of the theorem.

We now note that theorems 2 and 3 do not hold without the condition that $d(Sx,Ty) = 0$ if $d(x, Sx) + d(y,Ty) = 0$. This is easily seen by letting X be any complete metric space with at least two points and letting $S = T$ be the identity mapping on X. Then $d(x,Sx) + d(y,Ty) = 0$ for all x, y in X and so it follows that theorems 2 and 3 cannot hold without this extra condition.

We finally prove the following theorem for compact metric spaces

<u>Theorem</u> 4. Let S and T be continuous mappings of the

compact metric space X into itself such that for all x, y in X, either

$$d(Sx,Ty) < \frac{2d(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$, or

$$d(Sx,Ty) = 0$$

otherwise. Then each of S and T has a unique fixed point and these points coincide.

<u>Proof</u>: Let us suppose that $d(x,Sx) + d(y,Ty) > 0$ for all x, y in X. Then it follows from the conditions of the theorem that we must have

$$2d(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx) > 0$$

for all x,y in X. Hence the function $f: X \times X \longrightarrow R^+$ defined by

$$f(x,y) = \frac{d(Sx,Ty)[d(x,Sx) + d(y,Ty)]}{2d(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}$$

is continuous and less than 1 on the compact metric space $X \times X$, This implies that there exists $c < 1$ such that $f(x,y) \leq$ $\leq c$ on $X \times X$. It follows that

$$d(Sx,Ty) \leq \frac{2cd(x,Sx)d(y,Ty) + bcd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

for all x, y in X and so by theorem 3 there exists x in X such that $x = Tx = Sx$, giving a contradiction. Hence we must have $d(x,Sx) + d(y,Ty) = 0$ for some x, y in X and so $x = Tx =$ $= Ty = y$. Thus x is a common fixed point of S and T.

The uniqueness of x follows easily. This completes the proof of the theorem.

For further results on two mappings S and T satisfying a rational inequality see [2] and [3].

Remarks. We finally note the following variations of theorems 2, 3 and 4 respectively

Theorem 2′. Let X be a set and $d: X \times X \to [0,\infty)$ a function such that $d(x,y) = d(y,x)$ for all x, y in X and $d(x,y) = 0$ if and only if $x = y$. Let S and T be mappings of X into itself such that for all x, y in X, either

$$(Sx,Ty) \leq \frac{cd(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$ and $0 \leq c \leq 1$, or

$$d(Sx,Ty) = 0$$

otherwise. Then S and T are identical constant mappings on X.

Theorem 3′. Let X and d be as in theorem 2′. Let S and T be mappings of X into itself such that for all x, y in X, either

$$d(Sx,Ty) \leq \frac{cd(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$ and $1 < c < 2$, or

$$d(Sx,Ty) = 0$$

otherwise. Assume that the following condition is also satisfied:

if $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \leq$

$\leq (c-1)d(x_{n-1},x_n)$ for all $n>1$, then there exists a point x in X such that $d(x_n,x) \to 0$ and $d(x_n,Tx) \to d(x,Tx)$ as $n \to \infty$. Then each of S and T has a unique fixed point and these points coincide. Further, if this point is z, then for each x in X, $d(y_n(x),z) \to 0$, where $y_1(x) = x$, $y_{2n}(x) = Ty_{2n-1}$ and $y_{2n+1}(x) = Sy_{2n}(x)$.

Theorem 4′. Let X be a compact topological space and let d be as in theorem 2′. Let S and T be mappings of X into itself such that

$$d(Sx,Ty) < \frac{2d(x,Sx)d(y,Ty) + bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)}$$

if $d(x,Sx) + d(y,Ty) \neq 0$, where $b \geq 0$, or

$$d(Sx,Ty) = 0$$

otherwise. Assume that the function f (see the proof of theorem 4) is upper-semicontinuous over its domain. Then each of S and T has a unique common fixed point and these points coincide.

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