Commentationes Mathematicae Universitatis Carolinae

János Gerlits; István Juhász

On left-separated compact spaces

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 1, 53--62

Persistent URL: `http://dml.cz/dmlcz/105832`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,1 (1978)

ON LEFT-SEPARATED COMPACT SPACES

J. GERLITS and I. JUHÁSZ, Budapest

Abstract: We show that a left-separated compact T_2 space is both scattered and sequential. As a consequence we prove that if every subspace of a regular space X has a compact subspace of countable character, then X is first countable on a dense open subset, thus generalizing results by Archangelskiĭ and Ismail.

Key words and phrases: Left-separated, compact, scattered, sequential semi-stratifiable spaces.

AMS: Primary 54D30, 54D55 Ref. Ž.: 3.961.1
Secondary 54A25

Let us begin by recalling that a space X is left (right) separated iff it has a well-ordering $\prec$ such that every initial segment of X under $\prec$ is closed (open). These notions have been mainly investigated in connection with the cardinal functions z(X) and h(X), i.e. hereditary α-separatedness and α-Lindelöfness (cf. e.g. [5]). Right separatedness however has been widely studied in disguise: indeed a space is right separated iff it is scattered. Though at first this might sound surprising, a moment's reflection shows that this is actually trivial.

On the other hand almost nothing was known about the class of left separated spaces. In this paper we propose to show that they are also worthy of independent study, by establishing some interesting, even surprising, topological properties of the compact left separated T_2 spaces.

Theorem 1. If X is countably compact, T_3, and left separated by the well-ordering $\prec$, then X is scattered (i.e. also right separated).

Proof. As the properties of X are inherited by closed subsets, clearly it suffices to show that X has an isolated point.

Suppose this is not true, and define by induction for each $n \in \omega$ a point $p_n \in X$ and its closed neighbourhood F_n as follows. Put $F_0 = X$ and p_0 be the $\prec$-minimal element of F_0. Assume that p_n and F_n have already been defined in such a way that p_n is the $\prec$-first member of F_n. Now since F_n has non-empty interior and X has no isolated points, we can pick a point $p_{n+1} \in F_n \setminus \{p_n\}$ and its closed neighbourhood F_{n+1} such that $p_{n+1} = \min_{\prec} F_{n+1}$ and $F_{n+1} \subset F_n$, using that X is T_3 and left separated. Clearly we have $p_0 \prec p_1 \prec \dots p_n \prec \dots$, moreover $F_0 \supset F_1 \supset \dots F_n \supset \dots$ with $p_n = \min_{\prec} F_n$ for each $n \in \omega$. However then the set $\{p_k : k \in \omega\}$ has no limit point in X, since no point following all the p_k in $\prec$ can be a limit point because $\prec$ left separates X, while if $p \prec p_n$, then $X \setminus F_n$ is a neighbourhood of p that has finite intersection with $\{p_k : k \in \omega\}$, namely a subset of $\{p_k : k < n\}$. But this contradicts the countably compactness of X.

A.V. Archangelskiĭ proved in [1], assuming CH, that if X is T_3 and hereditarily of point countable type (i.e.

every subspace of X has a cover by compacta of countable character) then the points of first countability contain a dense open set in X. M. Ismail has recently shown that CH is not really needed for this [4]. Our next result, which is an easy corollary to Theorem 1, generalizes this result.

Corollary. Suppose X is T_3 and every subspace of X has a compact subset of countable character. Then the points of first countability contain a dense open set in X.

Proof. It follows from arguments in [1] that it suffices to prove the existence of a point $p \in X$ with $\chi(p,X) \leq \omega$. Now let S be a left separated dense subspace of X and $K \subset S$ be compact with $\chi(K,S) \leq \omega$. Then K is (countably) compact, T_3, and left separated, hence it has an isolated point p by Theorem 1. Moreover it is easily seen by the regularity of X that $\chi(K,X) = \chi(K,S) \leq \omega$, hence as p is isolated in K we clearly have $\chi(p,X) \leq \omega$ as well.

Our next result shows that the left separated compact T_2 spaces form only a narrow subclass of the compact scattered spaces.

Theorem 2. If X is compact, T_2, and left separated, then X is sequential.

Proof. We will split the proof into two steps. In the first step we show that X has countable tightness, i.e. $t(X) \leq \omega$. We do this by induction on the order type of the left separating well-ordering $\prec$ of X. Thus assume that every proper initial segment of X under $\prec$ has countable tightness and show that then so does X.

Assume, on the contrary, that $t(X) > \omega$. By [1] then X contains an uncountable free sequence $\langle p_\xi : \xi \in \omega_1 \rangle$. By Theorem 1 X is scattered, hence by [6] it is chain compact. Thus we may assume that the ω_1-sequence $\langle p_\xi : : \xi \in \omega_1 \rangle$ converges to a point $p \in X$. By [3] we can also assume that $\xi < \eta$ implies $p_\xi \prec p_\eta$, finally since X is left separated by $\prec$, it can be assumed that $p \prec p_0$.

Let $\xi \in \omega_1$ be a limit ordinal. As X is sequentially compact by [6], we can clearly select an ω-sequence of ordinals $\langle \xi_n : n \in \omega \rangle$, which increasingly converges to ξ and is such that the sequence $\langle p_{\xi_n} : n \in \omega \rangle$ converges in X, say to the point q_ξ. Since the sequence $\langle p_{\xi_n} : n \in \in \omega \}$ is $\prec$-increasing we clearly have $q_\xi \prec p_{\xi_n}$ for some $n \in \omega$. Let us define $f(\xi)$ as the smallest ξ_n, for which we have $q_\xi \prec p_{\xi_n}$.

Then f is a regressive function on the set of all limit ordinals, hence by Neumer's theorem there is an uncountable set $a \subset \omega_1$ on which f takes the same value, say η. Let Y denote the proper initital segment of X consisting of all $x \in X$ with $x \prec p_\eta$. Then $t(Y) \leq \omega$ and $p \in Y$.

For each $\xi \in \omega_1$ put $F_\xi = c\ell_X(\{p_\nu : \nu < \xi\})$; since the sequence $\langle p_\xi : \xi \in \omega_1 \rangle$ is free, we have $p \notin F_\xi$ for each $\xi \in \omega_1$. Now the sets $Y \cap F_\xi$ are closed in Y and clearly they are increasing, hence $t(Y) \leq \omega$ implies that $F = \cup \{Y \cap F_\xi : \xi \in \omega_1\}$ is closed in Y, and hence in X as well. Thus p has a closed neighbourhood U in X with $U \cap F = = \emptyset$. But $p_\nu \longrightarrow p$ (as $\nu \longrightarrow \omega_1$), hence there is $\mu \in \omega_1$ such that $p_\nu \in U$ for every $\mu \leq \nu < \omega_1$. Now let $\xi \in a$, $\xi > \mu$. Then we also have $\xi_n > \mu$ for all but finitely many $n \in \omega$, and thus since U is closed and $p_{\xi n} \longrightarrow q_\xi$ $(n \longrightarrow \omega)$, we have $q_\xi \in U$ as well.

However $\xi \in a$ implies that $f(\xi) = \eta$, and thus $q_\xi \preccurlyeq p_\eta$, i.e. $q_\xi \in Y$. Consequently $q_\xi \in F_\xi \cap Y \subset F$, which contradicts $U \cap F = \emptyset$.

Now let us turn to the second step of our proof. Again by induction on the order type of the left separating well-ordering $\preccurlyeq$ we show that X is sequential, i.e. every sequentially closed set is closed in X. Suppose that every proper initial segment of X under $\preccurlyeq$ is sequential and let $A \subset X$ be sequentially closed. If the order type of X with $\preccurlyeq$ is not a limit ordinal, i.e. X has a $\preccurlyeq$-last member, then trivially A is closed by the inductive hypothesis.

Next, if this order type is cofinal with ω, then let $\langle p_n : n \in \omega \rangle$ be a $\preccurlyeq$-cofinal ω-sequence in X, and put $Y_n = \{x \in X : x \preccurlyeq p_n\}$. Then for each $n \in \omega$ the set $A \cap Y_n$ is sequentially closed, hence closed in Y_n, hence compact, consequently A is σ-compact. On the other hand, if $\{q_n : n \in \in \omega\}$ is any countably infinite subset of A, then again by [6] there is a convergent (in X) subsequence of $\langle q_n : n \in \in \omega \rangle$ whose limit, by the sequential closedness of A, must be in A. But then A is also countably compact, which together with σ-compactness implies that it is compact and therefore closed in X.

Finally it remains to check the case in which the order type of X under $\preccurlyeq$ is greater than ω. To see that A is closed, let $p \in \overline{A}$ be arbitrary. Since $t(X) \leq \omega$ has been established already, we have a set $B \subset A$, $|B| \leq \omega$, such that

$p \in \overline{B}$. But now B cannot be cofinal in X, hence there is a proper initial segment Y of X with $p \in Y$ and $B \subset Y$. Since Y is closed, the set $A \cap Y$ is also sequentially closed, and thus it is also closed by the inductive hypothesis. Consequently we have $p \in \overline{B} \subset A \cap Y \subset A$, which shows that A is indeed closed.

Now that we have some information about the topological structure of left separated compact T_2 spaces, it is natural to strive for an internal, topological characterization of their class. Let us denote this class by $\mathcal{LC}$. According to our above results every $X \in \mathcal{LC}$ possesses the following three properties:

Ⓐ X is scattered;

Ⓑ X is sequential;

Ⓒ if $Y \subset X$ is arbitrary, then there is a discrete subspace $D \subset Y$ with $|D| = |Y|$.

Ⓒ follows from the fact that X is both right and left separated. These three properties however do not suffice to yield the desired characterization, as follows from our next result.

Theorem 3. Let $\langle T, \lhd \rangle$ be a Suslin tree with the tree topology (cf. [7]), and X be its one-point compactification. Then X satisfies conditions Ⓐ , Ⓑ and Ⓒ , but it is not left separated.

Proof. Ⓐ is trivial, Ⓑ - in fact that the one-point compactification of any Aronszajn tree is Fréchet-Uryson - is folklore. As to Ⓒ , it is easy to see that every $Y \subset X$ has $|Y|$ many isolated points.

Now assume, indirectly, that X is left separated by $\prec$. For every limit ordinal $\xi \in \omega_1$ pick a member $t_\xi \in T_\xi$ (the ξ^{th} level of T). Since $\prec$ left separates X we can choose for every limit ξ an $s_\xi \in T$ with $s_\xi \lhd t_\xi$ such that $s_\xi \lhd r \lhd t_\xi$ implies $t_\xi \prec r$. Now define $f(\xi)$ by $s_\xi \in T_{f(\xi)}$.

Then f is regressive, hence there is $a \subset \omega_1$, $|a| = \omega_1$, such that f takes the same value on a. Using [3] again we can also assume that $\xi, \zeta \in a$ and $\xi < \zeta$ imply $t_\xi \prec t_\zeta$. But then for any such ξ and ζ the elements t_ξ and t_ζ must be incomparable in $\langle T, \lhd \rangle$, since otherwise we had $s_\zeta \lhd t_\xi \lhd t_\zeta$ and therefore $t_\zeta \prec t_\xi$, a contradiction. This however means that $\{ t_\xi : \xi \in a \}$ is an uncountable antichain in $\langle T, \lhd \rangle$, which is impossible.

Next we formulate a condition which is sufficient for any space to be left separated. For this, however, we need a definition. Let X be any space. A sequence $\langle D_\nu : \nu < \varrho \rangle$ of disjoint subsets of X is called a vanishing sequence, if $X = \cup \{ D_\nu : \nu < \varrho \}$, moreover for each $\mu < \varrho$ the set D_μ is closed discrete in the subspace $\cup \{ D_\nu : \mu \le \nu < \varrho \}$. The ordinal ϱ is called the length of the vanishing sequence.

It is now quite easy to see that every space admitting a vanishing sequence of length $\le \omega$ is left separated. It came as a surprise to us, however, that every compact left separated space that we could think of happened to have a vanishing sequence of length $\le \omega$. This made us raise the following

<u>Problem</u>. Does every compact left separated space have a vanishing sequence of length $\le \omega$?

In fact we do not even know of a regular space that is both right and left separated, but is not the union of countably many discrete subspaces. On the other hand we can prove the following result.

<u>Theorem 4</u>. A semi-stratifiable space X is left separated if and only if it is the union of countably many closed discrete subspaces.

<u>Proof</u>. The "if" part is obvious. To see the converse, let $\prec$ left separate X and using [2] consider a family $\{U(n,x): n \in \omega, x \in X\}$ of open subsets of X such that

(i) $x \in U(n,x)$ for every $x \in X$;

(ii) if $x \in U(n,x_n)$ for every $n \in \omega$, then $x_n \longrightarrow x$.

Obviously, we can also assume that

(iii) $U(n,x) \subset \{y \in X: x \preceq y\}$ holds for each $x \in X$ and $n \in \omega$.

Now we claim that for each $x \in X$ there is $n(x) \in \omega$ such that if $x \in U(n(x),y)$, then $y = x$. Indeed, assume that for every $n \in \omega$ there is $y_n \neq x$ (and thus by (iii) $y_n \prec x$) such that $x \in U(n,y_n)$. Then by (ii) $y_n \longrightarrow x$, which is impossible as $\prec$ left separates X. Now put $A_n = \{x \in X: n(x) = n\}$. We claim that A_n is closed discrete in X. Indeed, if $z \in X$ is arbitrary, then $x \in U(n,z) \cap A_n$ implies $n(x) = n$ and therefore $x = z$, i.e. z has a neighbourhood that contains at most one member of A_n. This completes the proof.

Added in proof:

1) The following result obtained after the paper was submitted is of interest: If X is well-ordered by $\prec$ in such a way that every initial segment of X under $\prec$ is countably compact, then X is compact.

2) After having completed this paper we received the following paper which deals with similar topics, and some of whose results overlap with ours:

A.V. ARCHANGEL´SKIJ, O prostranstvach rastjanutych vlevo, Vestn. Moskov. Univ. (1977), 5, 30-36.

References:

[1] A.V. ARHANGELSKIĬ: On hereditary properties, General Top. Appl. 3(1973), 39-46.

[2] G. CREEDE: Concerning semi-stratifiable spaces, Pacific J. Math. 32(1970), 47-54.

[3] B. DUSHNIK and E.W. MILLER: Partially ordered sets, Amer. J. Math. 63(1941), 600-610.

[4] M. ISMAIL: On a theorem of Arhangelskiĭ, preprint.

[5] I. JUHÁSZ: Cardinal functions, Math. Centre Tract No 34, Amsterdam, 1971.

[6] S. MROWKA, M. RAJAGOPALAN, T. SUNDARARAJAN: A characterization of compact scattered spaces ..., Proc. TOPO 72, Springer Lecture Notes, Berlin, 1974, 288-298.

[7] M.E. RUDIN: Lectures on set theoretic topology, AMS Regional Conf. Series in Math. No 23, 1975.

Mathematical Institute
Hungarian Academy of Sciences
Reáltanoda u. 13-15, 1053 Budapest
Hungary

(Oblatum 5.12. 1977)