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ON CERTAIN CONGRUENCE LATTICES OF FINITE UNARY ALGEBRAS

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Abstract: This note contains the proof of the following statement: A finite unary algebra having not small congruence lattice of height two has no proper subalgebra. The methods used in the proof are entirely elementary.

Key words: Congruence lattice, unary algebra.

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There is a famous problem in Birkhoff's book [1, Problem 50, p. 110] about the representability of complete lattices as congruence lattices of algebras. This problem was solved by G. Grätzer and E.T. Schmidt [4], namely any algebraic lattice is isomorphic to a congruence lattice. However both the original proof and the new one (Pudlák and Tůma [5]) yield an infinite algebra with the given congruence lattice, even for finite lattices. So it is natural to ask, which finite lattices are isomorphic to congruence lattices of finite algebras (see [3, Problem 13, p. 116]). The largest class of finite lattices for which this problem is solved positively is the class of finitely fermentable lattices [5].

Using two-dimensional vector spaces over finite fields we can represent the lattices M_n of height (length) two with n atoms, whenever n - 1 is a prime power. By a result of R.W. Quackenbush [6] for other n's M_n cannot be represented as Con ($\mathcal{A}$) of a finite algebra $\mathcal{A}$ generating a congruence permutable variety. P. Goralčík [2] pointed out that M_7 - the smallest among such lattices - might prove crucial to the problem.

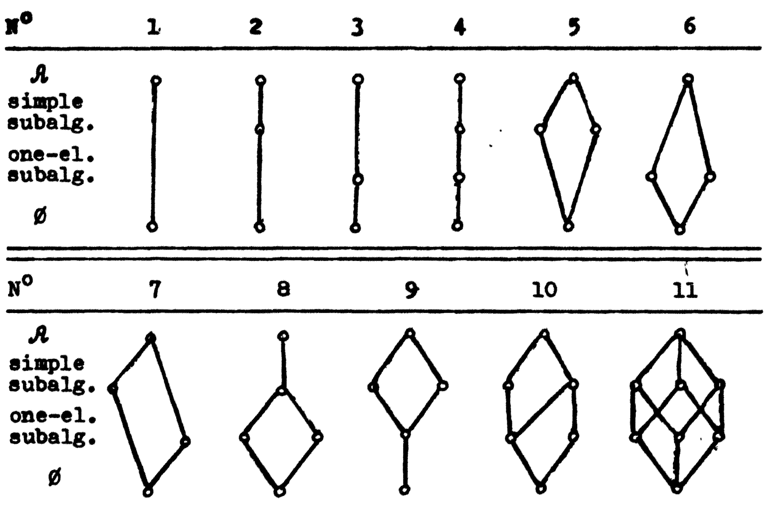
Investigating congruence lattices we may restrict ourselves to unary algebras. In the present paper we deal with finite unary algebras with congruence lattices isomorphic to M_n.

Proposition. Let $\mathcal{A}$ be a finite unary algebra, and suppose that the height of Con ($\mathcal{A}$) is 2. Then either the number of nontrivial congruences is less than 4, or $\mathcal{A}$ has no proper subalgebra.

Proof. Since $\mathcal{A}$ is unary, the join and meet of subalgebras are the set-theoretic ones. So Sub ($\mathcal{A}$) is distributive.

If $\mathcal{B} \subset \mathcal{A}$ and $\theta \in$ Con ($\mathcal{B}$), then define θ^* in the following way: $x \equiv y(\theta^*)$ iff either $x,y \in \mathcal{B}$ and $x \equiv y(\theta)$, or $x = y$. Now θ^* is a congruence on $\mathcal{A}$, because $\mathcal{A}$ is unary. If $\mathcal{A} \supset \mathcal{A}_1 \supset \dots \supset \mathcal{A}_n$ is a chain of subalgebras, then $\iota > \iota^*_{\mathcal{A}_1} > \dots > \iota^*_{\mathcal{A}_n}$ is a chain of congruences on $\mathcal{A}$. So from the condition on the height of Con ($\mathcal{A}$) we obtain that any chain in Sub ($\mathcal{A}$) has the form $\mathcal{A} \supset \mathcal{S} \supset \mathcal{O} \supset \emptyset$ (some members may be missing), where $\mathcal{S}$ is a simple, $\mathcal{O}$ is a one-element subalgebra.

From the previous two remarks it follows that Sub ($\mathcal{A}$) is one of the listed below lattices.



We must show that in cases (2) - (11) either the number of nontrivial congruences on $\mathcal{A}$ is less than 4, or the height of Con ($\mathcal{A}$) cannot be 2. Let M denote the monoid of all polynomials of $\mathcal{A}$.

Case (2). $\mathcal{S}$ is the only proper subalgebra of $\mathcal{A}$, $\mathcal{S}$ is simple. Let $\mathcal{T} = \mathcal{A} \setminus \mathcal{S}$ and for $x \in \mathcal{A}$ let $N(x) = \{f \in M \mid f(x) \in \mathcal{S}\}$, and define the equivalence ν as follows: $x \equiv y(\nu)$ iff $N(x) = N(y)$. It is easy to see that $\nu \in \mathrm{Con}\,(\mathcal{A})$, $\iota_{\mathcal{A}} > \nu \geqq \iota^*_{\mathcal{S}}$. Thus $\nu = \iota^*_{\mathcal{S}}$, that is for any pair of elements $u,v \in \mathcal{S}$, there exists an $f \in M$ such that either $f(u) \in \mathcal{T}$ and $f(v) \in \mathcal{T}$ or reversely. Choose an $g \in M$ in such a way that $|\mathrm{Im}\, g \cap \mathcal{T}|$ be the smallest possible positive number. ($\mathcal{A}$ is finite!) If $|\mathrm{Im}\, g \cap \mathcal{T}| > 1$, then we should apply the previous observation for two elements of $\mathrm{Im}\, g \cap \mathcal{T}$ to obtain an $f \in M$ satisfying $1 \leqq |\mathrm{Im}\, fg \cap \mathcal{T}| < |\mathrm{Im}\, g \cap \mathcal{T}|$. Thus the only possibility is that

$|\mathrm{Im}\, g \cap \mathcal{T}| = 1$. Let x be such an element for which $g(x) \in \mathcal{T}$. Since g(x) generates $\mathcal{A}$, there is an $h \in M$ such that $hg(x) = x$. Denote hg by m. We know that $m(\mathcal{A}) \subseteq \mathcal{S} \cup \{x\}$ and $m(x) = x$.

Let θ be an arbitrary nontrivial congruence on $\mathcal{A}$ differing from $\iota^*_{\mathcal{S}}$. Since $\theta \wedge \iota^*_{\mathcal{S}} = \omega$, each class of θ contains at most one element of $\mathcal{S}$. On the other hand as $\theta \vee \iota^*_{\mathcal{S}} = \iota$, each class of θ must contain an element of $\mathcal{S}$. Therefore such a congruence can be described by a function say $r: \mathcal{A} \longrightarrow \mathcal{S}$ which maps an element of $\mathcal{A}$ to the uniquely determined element of $\mathcal{S}$ belonging to the same congruence class of θ.

Let r_j $(1 \leq j \leq k)$ be the functions corresponding to the congruences θ_j. Suppose that $k \geq 3$. Since $\theta_1 \vee \theta_2 = \iota$, for any two elements $y,z \in \mathcal{S}$ there exist elements $s_0 = y$, $s_1, \dots, s_p = z$ and $t_1, \dots, t_p$ such that $s_i \in \mathcal{S}$ $(0 \leq i \leq p)$, $t_i \in \mathcal{T}$ $(1 \leq i \leq p)$ and $\{s_{i-1}, s_i\} = \{r_1(t_i), r_2(t_i)\}$ for all $1 \leq i \leq p$. Applying m we get that either $\{m(s_{i-1}), m(s_i)\} = \{m(t_i)\}$ (in the case when $m(t_i) \in \mathcal{S}$) or $\{m(s_{i-1}), m(s_i)\} = \{r_1(x), r_2(x)\}$ (when $m(t_i) = x$). Take $y = r_1(x)$ and $z = r_3(x)$. Then from this considerations it follows that $m(z) \in \{r_1(x), r_2(x)\}$, which contradicts to the conditions $\theta_3 \wedge \theta_1 = \theta_3 \wedge \theta_2 = \omega$. Therefore in the case (2) the number of nontrivial congruences does not exceed 3.

Case (3). $\{\sigma\}$ is the only proper subalgebra of $\mathcal{A}$. Let for $x \in \mathcal{A}$, $N(x) = \{f \in M \mid f(x) = \sigma\}$, and define ν by $x \equiv y(\nu)$ iff $N(x) = N(y)$. Obviously $\nu \in \mathrm{Con}(\mathcal{A})$, $[\sigma]\nu = \{\sigma\}$, and if for a $\theta \in \mathrm{Con}(\mathcal{A})$ $[\sigma]\theta = \{\sigma\}$, then $\theta \not\leq \nu$. On the other hand if $[\sigma]\theta \supsetneqq \{\sigma\}$, then $\theta = \iota$. So ν is the unique nontrivial congruence on $\mathcal{A}$.

Case (4). The proper subalgebras of $\mathcal{A}$ are $\mathcal{S}$ and $\{\sigma\}$, $\mathcal{S}$ is simple, $\sigma \in \mathcal{S}$. Suppose that $\theta_1, \theta_2, \iota^*_{\mathcal{S}}$ are different nontrivial congruences. If $x \equiv \sigma(\theta_i)$, then $[\sigma]\theta_i \supseteq [x]$, so, since $\theta_i \wedge \iota^*_{\mathcal{S}} = \omega$, $x = \sigma$. This implies $[\sigma](\theta_1 \vee \theta_2) = \{\sigma\}$ contradicting to $\theta_1 \vee \theta_2 = \iota$. Thus in case (4) the number of nontrivial congruences is at most 2.

Case (5). $\mathcal{S}_1$ and $\mathcal{S}_2$ are the only proper subalgebras of $\mathcal{A}$, they are simple and $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$. In this case $\omega < \iota^*_{\mathcal{S}_1} < \iota^*_{\mathcal{S}_1} \vee \iota^*_{\mathcal{S}_2} < \iota$ so the height of Con $(\mathcal{A})$ is at least 3, contrary to our hypothesis.

Case (6). In this case $\mathcal{A} = \{\sigma_1, \sigma_2\}$ so $\mathcal{A}$ is simple, which is not the case.

Case (7). $\mathcal{A} = \mathcal{S} \cup \{\sigma\}$, $\mathcal{S}$ is simple and has no proper subalgebra. It is easy to see that Con $(\mathcal{A})$ consists of ι, $\iota^*_{\mathcal{S}}$, ω .

Case (8). The only proper subalgebras of $\mathcal{A}$ are $\{\sigma_1\}$, $\{\sigma_2\}$ and $\{\sigma_1, \sigma_2\} = \mathcal{S}$. Suppose that $\theta \in$ Con $(\mathcal{A})$ is different from ι , $\iota^*_{\mathcal{S}}$, ω . Since $\theta \vee \iota^*_{\mathcal{S}} = \iota$, one of the σ_i's is congruent to an element of $\mathcal{A} \setminus \mathcal{S}$ by θ , and therefore $\theta = \iota$, because elements outside of $\mathcal{S}$ generate $\mathcal{A}$, and the σ_i's are fixed points. So in the case (8) the only nontrivial congruence is $\iota^*_{\mathcal{S}}$.

Case (9). $\mathcal{S}_1$ and $\mathcal{S}_2$ are simple subalgebras of $\mathcal{A}$, $\{\sigma\} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{A} = \mathcal{S}_1 \cup \mathcal{S}_2$, and there is no other subalgebra of $\mathcal{A}$. Let $\theta \in$ Con $(\mathcal{A})$ be different from ι, $\iota^*_{\mathcal{S}_1}$, $\iota^*_{\mathcal{S}_2}$, ω, then from the relations $\theta \wedge \iota^*_{\mathcal{S}_j} = \omega$ $\theta \vee \iota^*_{\mathcal{S}_j} = \iota$ $(j = 1, 2)$ it follows that θ is a matching between $\mathcal{S}_1$ and $\mathcal{S}_2$. Since θ is a congruence it yields an isomorphism between $\mathcal{S}_1$ and $\mathcal{S}_2$. If they are not isomorphic, then such θ cannot exist, so then the number of nontrivial congruences is 2. So suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are isomorphic. The number of isomorphisms between $\mathcal{S}_1$ and $\mathcal{S}_2$ is the same as the number of automorphisms of $\mathcal{S}_1$. For unary algebras the orbits of the automorphism group form a congruence. Now in our case σ is the only fixed point in $\mathcal{S}_1$, so $\{\sigma\}$ is an orbit of Aut $(\mathcal{S}_1)$, therefore - by the simplicity of $\mathcal{S}_1$, Aut $(\mathcal{S}_1) = \{1_{\mathcal{S}_1}\}$. So the number of nontrivial congruences is at most 3 in the case (9).

Case (10). The proper subalgebras of $\mathcal{A}$ are $\mathcal{S}$, $\{\sigma_1\}$, $\{\sigma_2\}$, $\{\sigma_1, \sigma_2\}$, $\sigma_1 \in \mathcal{S}$, $\sigma_2 \notin \mathcal{S}$, $\mathcal{S}$ is simple $\mathcal{A} = \mathcal{S} \cup \{\sigma_2\}$. One can easily check that in this case the only possible congruences are ι, $\iota^*_{\{\sigma_1,\sigma_2\}}$, $\iota^*_{\mathcal{S}}$, ω .

Case (11). In this case $\mathcal{A}$ is a three element set with identical operation, so Con $(\mathcal{A})$ coincides with the partition lattice of a three element set which contains 3 nontrivial partitions.

Thus the proof is complete.

References

[1] G. BIRKHOFF: Lattice theory, Amer. Math. Soc. publications, 2nd edition, 1948.

[2] P. GORALČÍK: Problem, Coll. Math. Soc. J. Bolyai 17,

Contributions to Universal Algebra, North-Holland, 1977, p. 604.

[3] G. GRÄTZER: Universal algebra, van Nostrand, 1968.

[4] G. GRÄTZER - E.T. SCHMIDT: Characterizations of congruence lattices of abstract algebras, Acta Sci. Math. (Szeged) 24(1963), 34-59.

[5] P. PUDLÁK - J. TŮMA: Yeast graphs and fermentation of algebraic lattices, Coll. Math. Soc. J. Bolyai 14, Lattice Theory, North-Holland, 1976, 301-341.

[6] R.W. QUACKENBUSH: A note on a problem of Goralčík, Coll. Math. Soc. J. Bolyai 17, Contributions to Universal Algebra, North-Holland, 1977, 363-364.

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