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Closure conditions in the nets

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Abstract: The existence of the sum (product) of permutations of the coordinate algebra of a net with singular points on one line is equivalent to the diagonal (Heidemeyer) closure condition of certain type. Some other closure conditions were found equivalent to the group properties of the operations mentioned.

Key words: Net, coordinate algebra, closure conditions.

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Introduction. In the theory of projective planes a lot is known about the connections between a validity of certain geometric conditions (frequently called closure conditions) and validity of certain algebraic conditions in the domain of coordinates of these projective planes or the existence of some types of automorphisms. In the theory of nets no such similar systematical theory is established hitherto, although there have been published numerous results having the character of one sided implication rather than systematical equivalence. Inspired by V. Havel's papers [1],[2],[3] I was looking for the simplest geometrical conditions such that algebraical conditions equivalent to them would form, in the whole, the known algebraic structures. Some examples will be shown in the author's next article.
I am indebted to V. Havel for calling my attention to this subject and for his permanent help and encouragement during the preparation of the present paper.

Part 1. Definitions

Definition: Under a net we shall understand a triple \((P, \mathcal{L}, (V_L)_{\ell \in \mathcal{L}})\), where \(P\) and \(I\) are non-void sets, \(\mathcal{L}\) is a set of some at least two-element subsets of \(P\) and \(I \rightarrow V_L\) is an injective mapping of \(I\) into \(P\) such that

1. \(v := \{ V_L \mid \ell \in I \} \in \mathcal{L} \)
2. \(\forall P \in P \setminus v \ \forall L \in I \ \exists ! \ell \in \mathcal{L} \ P, V_L \in \ell \)
3. \(#(a \cap b) = 1 \ \forall a, b \in \mathcal{L}, \ a \neq b. \)

The elements of \(P\) are called points, the elements of \(\mathcal{L}\) are called lines, the points \(V_L, \ell \in I\) are said to be singular or improper, the points of \(P \setminus v\) are said to be proper; the line \(v\) is said to be singular or improper; the lines of \(\mathcal{L} \setminus \{v\}\) are said to be proper; \#I is called degree of the net, \(#(\ell \setminus v)\) (is the same for all \(\ell \in \mathcal{L} \setminus \{v\}\)) is called order of the given net. The net \((v, \ell \setminus v, (V_L)_{\ell \in \mathcal{L}})\) is said to be trivial.

In the sequel, we shall restrict ourselves to the study of nets with degree \(\geq 3\). If \(A_1, A_2, \ldots, A_n \in L\) hold for \(A_1, A_2, \ldots, A_n \in P, \ell \in \mathcal{L}\), we shall write \(A_1 A_2 \ldots A_n\); for \(n = 2, A_1 \cap A_2\) we denote \(\ell := A_1 A_2\) and call \(\ell\) the join line of \(A_1, A_2\). If \(P \cap a, b\) holds for a distinct \(a, b \in \mathcal{L}\) and for \(P \in P\), then we shall write \(P := a \cap b\) and we call \(P\) the intersection point of \(a, b\). We shall write \(X \in N\) instead of \(X \in \mathcal{L} \setminus v, P \in N\) instead of \(P \in \mathcal{L} \setminus \{v\}, V_L \in N\) instead
Definition: By frame of a net $\mathcal{N}$ we mean a quadruple $(0, \alpha, \beta, \gamma)$ where $0$ is a proper point and $\alpha, \beta, \gamma \in I$, $\alpha \neq \beta \neq \gamma \neq \alpha$.

Definition: Under an admissible algebra we shall understand a quadruple $(S, 0, (\sigma_\ell)_{\ell \in J}, (\sigma_\ell^+)_{\ell \in J})$, where $S$ is a non-empty set, $0$ is a distinguished element of $S$, $J$ is an index set with one prominent index $\Theta$, $\sigma_\ell$ is a permutation of $S$ with $0 \sigma_\ell = 0$ and $\sigma_\ell^+$ is a loop operation over $S$ with neutral element $0$ such that

(i) $\sigma_\Theta = \text{id}_S$

(ii) $\forall \xi, \eta \in J, \xi \neq \eta, \forall b, c \in S \exists ! a \in S$

\[ a \sigma_\xi^+ \xi c = a \sigma_\eta^+ \eta c. \]

Construction 1. Let $\mathcal{N}$ be a net with a frame $(0, \alpha, \beta, \gamma)$ then define an admissible algebra in such a way that:

$S := 0 \vee_\infty \setminus \{ \emptyset \}$; $J := I \setminus \{ \alpha, \beta \}$; $\Theta := \gamma$; $\sigma_\ell : S \rightarrow S$;

$x \mapsto ((x \sigma_\beta \cap 0 \vee_\gamma) \vee_\infty \cap 0 \vee_\gamma \sigma_\beta \cap 0 \vee_\infty \forall \ell \in J$;

\[ a \sigma_\ell^+ \beta b = ((a \sigma_\beta \cap 0 \vee_\gamma) \vee_\infty \cap b \vee_\beta \vee_\infty \cap 0 \vee_\gamma \forall a, b \in S. \]

This admissible algebra is called the coordinate algebra of $\mathcal{N}$ with respect to the frame $(0, \alpha, \beta, \gamma)$. A map $S \times S \rightarrow P \setminus v$ defined by the rule $(a, b) \mapsto (a \sigma_\beta \cap 0 \vee_\gamma) \vee_\infty \cap (b \vee_\beta)$ is said to be the coordinate map.

Construction 2. Let $(S, 0, (\sigma_\ell)_{\ell \in J}, (\sigma_\ell^+)_{\ell \in J})$ be an admissible algebra $\star S > 1$, with a prominent index $\Theta$. Define $I := J \cup \{ \omega_1, \omega_2 \}$, where $\{ \omega_1, \omega_2 \} \cap J = \emptyset$, $\star \{ \omega_1, \omega_2 \} = 2$.
Then \((\mathcal{P}, \mathcal{L}, (V_{\leq})_{\mathcal{L} \in J})\) is a net called the net over admissible algebra \((S, 0, (\mathcal{S}_{\leq})_{\mathcal{L} \in J}, (+_{\leq})_{\mathcal{L} \in J})\).

The proof are omitted here.

**Part 2. Closure conditions**

**Definition:** Let \(\mathcal{N}\) be a net of degree \(\geq 4\) and \(\alpha, \beta, \gamma, \sigma\) be pairwise distinct elements of \(I\). By the Minor Desargues Condition (denoted by MDC) of type \((\alpha, \beta, \gamma, \sigma)\) in \(\mathcal{N}\) we mean the following implication: 
\[
(\forall A, B, C, A', B', C' \in \mathcal{P} \setminus v) \\
\left( AA'V_\sigma \land BB'V_\sigma \land CC'V_\sigma \land ABV_\gamma \land A'B'V_\gamma \land ACV_\beta \land A'C'V_\beta \land B'C'V_\alpha \rightarrow B'C'V_\alpha \right).
\]

If MDC of type \((\alpha, \beta, \gamma, \sigma)\) in \(\mathcal{N}\) holds for fixed \(\sigma \in I\) and all distinct \(\alpha, \beta, \gamma \neq \sigma\) we shall say that MDC of type \((\sigma)\) holds in \(\mathcal{N}\). If MDC of type \((\sigma)\) holds for all \(\sigma \in I\) in \(\mathcal{N}\) we shall say that MDC holds universally in \(\mathcal{N}\).

**Theorem 2.1.** Let \(\mathcal{N}\) be a net of degree \(\geq 4\) with the frame \((0, \alpha, \beta, \gamma)\) and let MDC of type \((\alpha)\) hold in \(\mathcal{N}\). Then in a coordinate algebra of \(\mathcal{N}\) with respect to the frame \((0, \alpha, \beta, \gamma), +_\gamma\) is a group operation and \(+_\gamma = _{\leq}\) for all \(\leq \in I \setminus \{\alpha, \beta\}\). Under the previous assumptions, MDC of type \((\beta)\) holds in the net \(\mathcal{N}\) if and only if \(\mathcal{S}_{\leq}\) is an automorphism of the group \((OV_{\alpha} \setminus \{V_{\alpha}\}; +_\gamma) = (S, +)\) for all...
\( \subseteq I \setminus \{ \alpha, \beta \} \).


Definition: We denote \( \Sigma = \{ (x_\alpha)_{\alpha \in J} \cup \{ \phi \} \} \), where the endomorphism \( \phi_\beta \) is defined by \( \phi_\beta = 0 \quad \forall \alpha \neq S \). The sum \( \phi_\phi + \phi_\mu \) is defined for \( \phi_\phi, \phi_\mu \in \Sigma \) by the rule
\[
\phi_\phi + \phi_\mu = \phi_\phi + \phi_\mu \quad \forall \alpha \in S.
\]

Lemma 2.2: For every \( \phi_\mu \in \Sigma \), \( \phi_\beta + \phi_\mu = \phi_\mu + \phi_\beta = \phi_\mu \).

The proof is trivial.

Definition: Let \( \mathcal{N} \) be a net of degree \( \geq 4 \), and \( \alpha, \beta, \gamma, \phi \) pairwise distinct indexes of \( I \). By diagonal condition (shortly DC) of type \( (\alpha, \beta, \mu, \phi) \) in \( \mathcal{N} \) we shall mean the following implication: \( (\forall A, B, C, D \in \mathcal{P} \setminus \varnothing) (AB_\beta \wedge CD_\beta \wedge AD_\alpha \wedge BC_\alpha \wedge AC_\mu \rightarrow BD_\phi) \).

![Diagram](image)

Proposition 2.3: Let \( \mathcal{N} \) be a net of degree \( \geq 4 \), with a frame \((0, \alpha, \beta, \gamma)\) and let MDC of type \((\alpha)\) hold in \( \mathcal{N} \). Then for \( \mu \in I \setminus \{ \alpha, \beta \} \) there is exactly one \( \phi \in I \setminus \{ \alpha, \beta \} \) such that \( x_\phi \phi = -x_\mu \mu \quad \forall x \in S \) iff DC of type \((\alpha, \beta, \mu, \phi)\),
restricted to $A = 0$, $A+B$, holds in $\mathcal{N}$.

**Proof:** Choose an element $0+a \in S$ and let $A = 0$, $B = (a,0)$, $C = (a,a^{\mu})$, $D = (0,a^{\alpha})$. These points are satisfying all assumptions of DC of type $(\alpha, \beta, \mu, \varphi)$ in $\mathcal{N}$ where $A = 0$ and $A+B$ hold. Then the conclusion of this DC $\mathcal{B}V_{\varphi}$ holds iff the point $B = (a,0)$ is an element of the line $n =: \mathcal{B}V_{\varphi}$ of the equation $y = x^{\varphi} + a^{\phi}$, i.e. iff $a^{\varphi} = -a^{\mu}$ $\forall a \in S$. If there is an index $\omega \neq \varphi$ such that $a^{\omega} = -a^{\mu}$, then the point $(a,0)$ is an element of the line $\mathcal{B}V_{\omega}$ of the equation $y = x^{\omega} + a^{\mu}$. Hence there is a line $m$, $B,D,V_{\omega}$ a $m$, $m+n$ which contradicts the definition of the net. (If $B = D$ then $a = 0$ i.e. $A = B$.) Thus there is exactly one index $\varphi$ of the previous property.

**Corollary.** Let $\mathcal{N}$ be a net of degree $\geq 4$ with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S,0,(\sigma_{\epsilon}),_{\epsilon \in J}, (+),_{\epsilon \in J})$ with respect to $(0, \alpha, \beta, \gamma)$, and let MDC of type $(\alpha)$ hold in $\mathcal{N}$. Then for each $\sigma_{0} \in \Sigma$ exactly one $\sigma_{\varphi} \in \Sigma$ exists such that

$$\sigma_{\varphi} + \sigma_{\omega} = \sigma_{\beta} = \sigma_{\mu} + \sigma_{\varphi},$$

iff DC of type $(\alpha, \beta, (\mu, \varphi)$ together with $A = 0$, $A+B$ holds in $\mathcal{N}$.

The proof follows from Proposition 2.3 and from the properties of the group $(S, +)$ (see [4] pp. 14 - 15).

**Remark.** The element $\sigma_{\varphi}$ is said to be opposite to the element $\sigma_{\mu}$; let us write $\sigma_{\varphi} =: -\sigma_{\mu}$; $a^{\varphi} = a^{\varphi} + a^{\mu} = -a^{\mu} + a^{\mu} = 0$ holds provided that the element $-\sigma_{\mu} \in \Sigma$ exists.
Definitions: Let $N$ be a net of degree $\geq 5$; let $\alpha, \beta \in I$, $\alpha \neq \beta$; $\varphi, \lambda, \tau \in I$ all distinct from $\alpha, \beta$. Then the Generalized diagonal condition (shortly GDC) of type $(\alpha, \beta, \varphi, \lambda, \tau)$ in $N$ is defined as the following implication: 

\[
( \forall A, B, C, D \in \mathcal{P} \setminus \varnothing ) (ABV_{\beta} \wedge ADV_{\varphi} \wedge BCV_{\lambda} \wedge ACV_{\tau} \wedge BDV_{\alpha} \implies CDV_{\tau}).
\]

Proposition 2.4: Let $N$ be a net of degree $\geq 5$ with a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma_{\lambda})_{\lambda \in J}, (t_{\lambda})_{\lambda \in J})$ with respect to this frame. Let MDC of type $(\alpha, \beta)$ hold in $N$. Then for all $\varphi, \lambda \in I \setminus \{ \alpha, \beta \}$ there exists exactly one $\tau \in I \setminus \{ \alpha, \beta \}$ such that

\[
\sigma_{\tau} = x \cdot \sigma_{\varphi} + x \cdot \sigma_{\lambda} \quad \forall x \in S
\]

iff GDC of type $(\alpha, \beta, \varphi, \lambda, \tau)$ with $A = 0, A \neq B$ holds in $N$.

Proof: Choose an element $0 \neq a \in S$ and let $A = 0, B = (a, 0)$. Then $C = (a, a \sigma_{\varphi}), D = (0, -a \sigma_{\lambda})$ (the points $D, B, V_{\lambda}$ are elements of a line of the equation $y = x \sigma_{\lambda} - a \sigma_{\lambda}$).

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and the line \( CV \) has the equation \( y = x^a - \alpha x + \alpha \).

These points satisfy all assumptions of GDC where \( A = 0, A \neq B \). The point \( D \) is an element of the line \( CV \) iff
\[
- \alpha^a = - \alpha \beta + \alpha \lambda \quad \text{i.e.} \quad \alpha^a = \alpha \beta + \alpha \lambda \quad \forall \alpha \in S.
\]

The assumption that an index \( \omega \neq \sigma \) exists such that \( \alpha^a = \alpha \beta + \alpha \lambda \) leads to the contradiction with the definition of the net.

**Corollary:** Let \( N \) be a net of degree \( \geq 5 \) with a frame \((0, 0, \beta, \gamma)\) and with a coordinate algebra \((S, 0, (\sigma^a)_{\alpha \in I}, (+)_{\alpha \in I})\) with respect to this frame. Let MDC of type \((\alpha)\) hold in \( N \). Then for all \( \sigma^a, \lambda \in \Sigma \) there exists exactly one \( \alpha \in I \) such that
\[
\sigma^a = \sigma^a + \sigma^a
\]
iff GDC of type \((\alpha, \beta, \phi, \lambda, \sigma)\) with \( A = 0, A \neq B \) holds in \( N \). The proof follows from Proposition 2.4 and from the definition of the operation \( + \) in \( \Sigma \).

**Definition:** Let \( N \) be a net of degree \( \geq 4 \), let \( \alpha, \beta \in I, \alpha \neq \beta \). If for each \( \mu \in I, \mu \neq \alpha, \mu \neq \beta \), a \( \lambda \in I \) exists such that DC of type \((\alpha, \beta, \mu, \lambda)\) holds in \( N \) then we say that DC of type \((\alpha, \beta)\) holds in \( N \).

Let \( N \) be a net of degree \( \geq 5 \), let \( \alpha, \beta \in I; \alpha \neq \beta \). If for all \( \phi, \lambda \in I \), distinct from \( \alpha, \beta \), a \( \Sigma \in I \) exists such that GDC of type \((\alpha, \beta, \phi, \lambda, \sigma)\) holds, then we say that GDC of type \((\alpha, \beta)\) holds in \( N \).

**Proposition 2.5:** Let \( N \) be a net of degree \( \geq 5 \) with
a frame $(0, \alpha, \beta, \gamma)$ and with a coordinate algebra 
$(S, 0, (\sigma_\mu)_{\mu \in J}, \langle \alpha, \beta, \gamma \rangle)$ with respect to this frame. Let MDC 
of type $(\alpha)$ hold in $N$. If we denote $\Xi = \{(\sigma_\mu)_{\mu \in J}\} \cup \{\sigma_\alpha\}$ (where $\sigma_\alpha := 0 \ \forall \alpha \in S$) and define 
$+: \sigma_\alpha + \sigma_\mu := \sigma_\beta + \sigma_\mu \ \forall \alpha \in S; \sigma_\alpha, \sigma_\mu \in \Xi,$
then DC of type $(\alpha, \beta)$ and GDC of type $(\alpha, \beta)$ together 
with $A \ast 0, A \neq B$ hold in $N$ iff $(\Xi, +)$ is a group.

The proof follows from Propositions 2.3 and 2.4. The 
validity of the associative law follows immediately.

**Definition:** Let $N$ be a net of degree $\geq 3$, $\alpha, \beta, \gamma, 
\mu, \varphi, \tau \in I$, $\alpha + \beta + \gamma \neq \alpha$. By the Reidemeister condi-
tion (denoted by RC) of type $(\alpha, \beta, \gamma, \mu, \varphi, \tau)$ in $N$ we
mean the implication:

$$(\forall A, B, C, D, H \in P \ \forall) \ (HAV_\mu \land HBV_\varphi \land HCV_\tau \land ABV_\beta \land DCV_\beta \land 
\land ADV_\mu \land BCV_\beta \implies HDV_\tau).$$

If RC of type $(\alpha, \beta, \gamma, \mu, \varphi, \tau)$ holds for fixed $\alpha, \beta, \gamma \in I$ and all $\mu, \varphi, \tau \in I$ we shall say that RC of 
type $(\alpha, \beta, \gamma)$ holds in $N$. 
Proposition 2.6: Let \( \mathcal{N} \) be a net of degree \( \geq 3 \) with a frame \((0, \alpha, \beta, \gamma)\) and a coordinate algebra \((S, \Omega, (\Sigma^\alpha, \Sigma^\beta, \Sigma^\gamma))\) with respect to this frame. Then for \( \omega, \varphi \in I \setminus \{\alpha, \beta, \gamma\}; \omega \neq \varphi \), exactly one \( \tau \in I \setminus \{\alpha, \beta\}\) exists such that
\[
(x^\varphi)^{\tau \varphi} = x^\varphi \quad \forall x \in S
\]
iff RC of type \((\alpha, \beta, \gamma)\) with \(H = 0\) holds in \( \mathcal{N} \).

Remark: We denote \( x^\varphi := (x^\varphi)^{\tau \varphi} \).

Proof: Choose an element \( x \in \Omega \setminus \{V, \Gamma\} \) and denote
\[
R := x \cap \Omega \setminus \{V, \Gamma\}, \quad A := R \cap \Omega \setminus \{V, \Gamma\}, \quad x^\varphi = A \cap \Omega \setminus \{V, \Gamma\}, \quad B := A \cap \Omega \setminus \{V, \Gamma\}, \quad C := B \cap \Omega \setminus \{V, \Gamma\}, \quad (x^\varphi)^{\tau \varphi} = C \cap \Omega \setminus \{V, \Gamma\}
\]
(such an element exists in accordance with the definition).
The mapping \( \sigma^\varphi : x \mapsto x^\varphi \) is a permutation provided that there is a point \( V^\varphi \) such that \( \overline{D} \) for \( D := (x^\varphi)^{\tau \varphi} \). But such a point \( V^\varphi \) exists iff RC of type \((\alpha, \beta, \gamma)\) with \(H = 0\) holds.

Proposition 2.7: Let \( \mathcal{N} \) be a net of degree \( \geq 3 \) with a frame \((0, \alpha, \beta, \gamma)\) and with a coordinate algebra \((S, \Omega, (\Sigma^\alpha, \Sigma^\beta, \Sigma^\gamma))\) with respect to this frame. Then for \( \omega \in I \setminus \{\alpha, \beta, \gamma\} \), exactly one index \( \varphi \in I \setminus \{\alpha, \beta, \gamma\} \) exists such that
\[
(x^\varphi)^{\tau \varphi} = (x^\varphi)^{\omega \varphi} = x \quad \forall x \in S
\]
iff RC of type \((\alpha, \beta, \gamma, \omega, \varphi, \gamma)\) with \(H = 0\) holds in \( \mathcal{N} \).

Proof: Choose an element \( x \in \Omega \setminus \{V, \Gamma\} \) and let
\[
A := x \cap \Omega \setminus \{V, \Gamma\}, \quad D := A \cap \Omega \setminus \{V, \Gamma\}
\]
then \( x^\varphi = D \cap \Omega \setminus \{V, \Gamma\} \). Let
C:= xCV_\beta \cap OV, B:= CV_\omega \cap OV, \text{ the point } (xCV_\omega)^CV_\beta = BV_\beta \cap OV, \text{ is equal to the point } x \text{ iff for the points } 0, A, B, C, D \text{ of } \mathcal{N}, RC \text{ of type } (\alpha, \beta, \gamma, \mu, \varphi, \gamma) \text{ holds. The second part of the proposition follows similarly.}

**Remark:** The permutation \( \sigma_\varphi \) from Proposition 2.7 is said to be inverse to \( \sigma_\mu \) and will be denoted by \( \sigma_\mu^{-1} \).

Clearly if \( \sigma_\varphi = \sigma_\mu^{-1} \) then \( \sigma_\mu = \sigma_\varphi^{-1} \).

**Corollary:** If for \( \mathcal{N} \) from Proposition 2.7 RC of type \( (\alpha, \beta, \gamma, \mu, \varphi, \gamma) \) with \( H = 0 \) holds for fixed \( \alpha, \beta, \gamma \in I \) and each \( \mu \) (resp. \( \varphi \) ), then for each permutation \( \sigma_\mu \) (resp. \( \sigma_\varphi \) ), exactly one inverse permutation \( \sigma_\mu^{-1} = \sigma_\varphi \) (resp. \( \sigma_\varphi^{-1} = \sigma_\mu \)) exists.

**Proof:** The existence of \( \sigma_\mu^{-1} \) follows from Proposition 2.7; the existence of \( \sigma_\varphi^{-1} \) follows analogously.

**Definition:** Let \( \mathcal{N} \) be a net of degree \( \geq 5 \) and let \( \alpha, \beta, \mu, \varphi, \lambda \) be pairwise distinct indexes from I. By the Pappus condition (shortly PC) of type \( (\alpha, \beta, \varphi, \mu, \lambda) \) in \( \mathcal{N} \) we mean the following implication:

\[
(\forall O, A, B, C, A', B', C' \in P \setminus v) (OABC_\omega \land OA'B'C'_{\beta} \land AA'B'_{\varphi} \land CC'B'_{\mu} \land AB'B'_{\lambda} \land BC'B'_{\varphi} \land A'B'_{\mu} \land B'C'_{\omega} \land A'BV_\alpha \implies B'CV_{\lambda}).
\]

If PC of type \( (\alpha, \beta, \varphi, \mu, \lambda) \) holds in \( \mathcal{N} \) for fixed \( \alpha, \beta, \mu \in I \) (resp. \( \alpha, \beta \in I \)) and for any \( \varphi, \lambda \in I \) (resp. \( \varphi, \mu, \lambda \in I \)), then we shall say that PC of type \( (\alpha, \beta, \mu) \) (resp. PC of type \( (\alpha, \beta) \)) holds in \( \mathcal{N} \).

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Proposition 2.8: Let $\mathcal{N}$ be a net of degree $\geq 8$, with a frame $(0', \alpha, \beta, \gamma)$ and with a coordinate algebra $(S, 0, (\sigma)_{\lambda \in \lambda}, (\omega)_{\lambda \in \lambda})$ with respect to this frame. Let MDC of type $(\alpha)$ and DC of type $(\alpha, \beta)$ with $A = 0'$, $A + B$ hold in $\mathcal{N}$.

Then for $\omega, \tau \in I \setminus \{\alpha, \beta\}$, $\omega = \tau$,

$$(x, \omega, \tau, \sigma) = (x, \tau, \omega, \sigma) \quad \forall x \in S \text{ holds},$$

iff PC of type $(\alpha, \beta, \omega)$ with $x, \omega, \tau, \sigma$ holds in $\mathcal{N}$.

Proof: Choose an element $a \in 0'V_{\omega} \setminus \{V_{\omega}\}$ and denote

$0' = 0', A := (0, a, \sigma), B := (0, a, \omega), C := (0, (a, \omega) \sigma),$

$D := (0, (a, \omega) \sigma, 0), A' := (a, 0), B' := (a, \omega, 0), C' := (a, \omega, 0).$

Then the line $AB'$ is expressed by the equation $y = x, \omega, \tau, \sigma$, where $0 = (a, \tau, \omega, \sigma)$ holds for $\sigma$. Similarly, the
line BC' can be expressed by the equation \( y = x^{BC'} + a \) with \( 0 = (a^{\omega})^{BC'} + a^{\omega} \). Both of these relations imply the condition \( x^{BC'} = x = x^{BC} \) \( \forall x \in S \), i.e., \( \xi_{\omega} = \xi_{\omega} \) and also \( AB^{\omega} \land BC^{\omega} \) hold in \( \mathcal{N} \). Analogously, for \( \xi_{\omega} \) the line AA' yields the condition \( 0 = a^{\omega} + a^{\omega} \) as well as CC' for \( \overline{\mathcal{P}} \). The condition \( 0 = (a^{\omega})^{\bar{A}} + (a^{\omega})^{\bar{A}} \) yields. These conditions imply \( x^{\bar{A}} = -x^{\bar{A}} = x^{\bar{A}} \) \( \forall x \in S \), i.e. \( \xi_{\omega} = \xi_{\omega} \) and \( AB^{\omega} \land BC^{\omega} \).

For \( \xi_{\omega} \) (resp. \( \xi_{\omega} \)), the line A'B (resp. B'\bar{C}) provides the condition \( 0 = a^{\omega} + a^{\omega} \) (resp. \( 0 = (a^{\omega})^{\bar{A}} + (a^{\omega})^{\bar{A}} \)). Hence \( x^{\bar{A}} = -x^{\bar{A}} = x^{\bar{A}} \) \( \forall x \in S \), i.e. \( \xi_{\omega} = \xi_{\omega} \) and \( AB^{\omega} \land BC^{\omega} \).

If \( (a^{\omega})^{\omega} = (a^{\omega})^{\omega} \) holds then \( 0 = 0 \) and PC of type \( (\gamma, \beta, \omega) \) with \( x^{\omega} = -x \) \( \forall x \in S \) holds in \( \mathcal{N} \).

Conversely, if PC of type \( (\gamma, \beta, \omega) \) with \( x^{\omega} = -x \) \( \forall x \in S \) holds in \( \mathcal{N} \), then \( 0 = 0 \) i.e. \( (a^{\omega})^{\omega} = (a^{\omega})^{\omega} \) for any element \( a \in S \).

**Proposition 2.9:** Let \( \mathcal{N} \) be a net of degree \( \geq 8 \) with a frame \((0', \alpha, \beta, \gamma)\) and a coordinate algebra \((S, O, (\xi_{\omega})_{\omega \in I}, (+_{\omega})_{\omega \in I})\) with respect to this frame. Let MDC of type \( (\gamma) \), DC of type \( (\alpha, \beta) \) with \( A = 0' \), \( A \Gamma B \) and RC of type \( (\alpha, \beta, \gamma) \) with \( H = 0' \) hold in \( \mathcal{N} \).

Then for all distinct \( \omega, \gamma, \omega, \omega \in I \backslash \{\alpha, \beta, \gamma\} \),

\[
(x^{\omega})^{(\omega)} = (x^{\omega})^{(\omega)} \quad \forall x \in S,
\]

\[
x^{\omega} = x^{\omega} \quad \forall x \in S
\]

hold iff PC of type \( (\alpha, \beta) \) with \( 0 = 0' \) holds in \( \mathcal{N} \).
Proof: The commutativity law for the permutations 
\{ (\sigma_v)_{v \in J} \} follows from PC, the existence of the product 
of permutations follows from RC (see Proposition 2.6). It 
is necessary to prove the associativity law for those per­mutations. Choose an element a \in V_\infty \setminus V_\alpha \setminus V_\beta and put 0:= 0', 
A:= (0, a\sigma_\alpha \sigma_\lambda), B:= (0, a\sigma_\alpha \sigma_\lambda), C:= (0, a\sigma_\alpha (\sigma_\omega \sigma_\lambda)), \bar{C}:= 
:= (0, a\sigma_\omega (\sigma_\tau \sigma_\lambda)), A' := (a\sigma_\lambda, 0), B' := (a\sigma_\lambda, 0), C' := 
:= (a\sigma_\omega \sigma_\lambda, 0).

From the equations of the lines AA' and CC' it follows 
0 = (a\sigma_\lambda)^T + a\sigma_\tau \sigma_\lambda = (a\sigma_\omega \sigma_\lambda)^T + a\sigma_\tau \sigma_\lambda \quad \text{and} 
0 = (a\sigma_\omega \sigma_\lambda)^T + a\sigma_\tau \sigma_\lambda \quad \text{i.e.} \quad \sigma_\lambda = \sigma_\tau + \sigma_\omega \sigma_\lambda.

From the equations of the lines AB' and BC' it follows 
0 = (a\sigma_\lambda)^T + a\sigma_\lambda \quad \text{and} 
0 = (a\sigma_\omega \sigma_\lambda)^T + a\sigma_\lambda \quad \text{i.e.}

\sigma_\lambda = \sigma_\omega \sigma_\lambda. \quad \text{Thus} \quad A'BV_\lambda \land A'CV_\lambda \quad \text{holds.}

The condition 

(\ast) \quad a\sigma_\tau (\sigma_\omega \sigma_\lambda) = a\sigma_\omega (\sigma_\tau \sigma_\lambda) \quad \forall a \in S

holds iff C = \bar{C} (see Proposition 2.8) which is equivalent to 
PC of type (\alpha, \beta) with 0 = 0'. The condition (\ast) holds 
iff \sigma_\tau (\sigma_\omega \sigma_\lambda) = \sigma_\omega (\sigma_\tau \sigma_\lambda) \quad \forall a \in S \quad \text{holds i.e.,} \quad \sigma_\tau (\sigma_\lambda \sigma_\omega) = 
= a(\sigma_\tau \sigma_\lambda)^T \quad \forall a \in S, \quad \text{which completes the proof.}

Proposition 2.10: Let \mathcal{N} be a net of degree \geq 8 to­gether with a frame (0', \alpha, \beta, \gamma) and a coordinate algebra 
(S, 0, (\sigma_v)_{v \in J}, (\tau_v)_{v \in J}, (\omega_v)_{v \in J}) associated with this frame. Let MDC 
of type (\alpha) and DC of type (\alpha, \beta) with A = 0', A*B hold
Then $\mathbb{Z}^*_n = \{ (\sigma_i)_{i \in J} \}$ with the product operation is a commutative group iff $RC$ of type $(\alpha, \beta, \gamma)$ with $H = 0'$ and $PC$ of type $(\alpha, \beta)$ with $0 = 0'$ hold in $N'$. The proof follows from Propositions 2.6, 2.7 and 2.8.

References


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