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Liouville formula for systems of linear homogeneous Itô stochastic differential equations

Ivo Vrkoč, Praha

Abstract: Let $X(t)$ be the fundamental matrix solution of Itô equation (1) and $D(T) = \det X(T)$. The process $D(t)$ is a solution of (2) and hence given by (6). It is shown that $X(t)$ is regular and a formula for solutions of nonhomogeneous linear Itô equations is derived.

Key words: Linear Itô stochastic equations, Liouville formula, fundamental matrix solutions, variation of constants formula.

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The general system of linear homogeneous Itô stochastic differential equations can be written in the vector form

$$\begin{align*}
\text{(1)} & \quad dx = A(t)x dt + \sum_{j=1}^{k} B^{(j)}(t)x dw_j, \\
\text{where } x & \text{ is an } n\text{-dimensional vector, } A(t), B^{(j)}(t), j = 1,\ldots, k, \text{ are matrix functions of the type } n \times n \text{ defined on } <0,\infty), \text{ } w_j(t) \text{ are stochastically independent Wiener processes.}
\end{align*}$$

Assume that $\|A(t)\|, \|B^{(j)}(t)\|$ are measurable and locally bounded on $<0,\infty)$. A matrix function $X(t)$ of the type $n \times n$ defined on $<t_0,\infty)$, $t_0 \geq 0$ is called a fundamental matrix solution of (1) if the columns of $X(t)$ are solu-

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tions of (1) on \( \langle t_0, \infty \rangle \) and \( X(t_0) \) is the unit matrix. The existence and unicity of the solutions of (1) is proved in [11, 12]. Denote \( D(t) = \det X(t) \).

**Theorem 1.** The process \( D(t) \) is a solution of

\[
\frac{dD}{dt} = \operatorname{tr} A(t) + \frac{1}{2} \sum_{p,q,j} \left( B_{pp}(j)(t)B_{qq}(j)(t) - B_{pq}(j)(t)B_{qp}(j)(t) \right) D(t) \, dt + D(t) \sum_{j} \operatorname{tr} B(j)(t) \, dw_j .
\]

**Proof.** Let \( x_{i}^{(j)}(t) \) be the \( i \)-th element of the \( j \)-th column of \( X(t) \). The determinant \( D(t) \) can be written by the well-known formula

\[
D(t) = \sum_{j_1, \ldots, j_n} \varepsilon(j_1, \ldots, j_n) x_{1}^{(j_1)}(t) \cdots x_{n}^{(j_n)}(t),
\]

where the indices \( j_1, \ldots, j_n \) assume the values of all permutations of \( 1, \ldots, n \), \( \varepsilon(j_1, \ldots, j_n) = 1 \) or \(-1\) if \( j_1, \ldots, j_n \) is an even or an odd permutation, respectively. Applying the Itô formula to (3) we obtain

\[
\frac{dD}{dt} = \sum_{j_1, \ldots, j_n} \varepsilon(j_1, \ldots, j_n) \left[ \sum_{p=1}^{n} x_{p}^{(j_p)} \cdots x_{p-1}^{(j_{p-1})} \right] dx_{p}^{(j_{p+1})} \cdots dx_{p-1}^{(j_{p-2})}
\]

and due to (1) we obtain

\[
dx_{p}^{(j_{p+1})} dx_{q}^{(j_{q+1})} = \sum_{j} \left( \frac{1}{2} \sum_{k} B_{pk}(j) x_{k}^{(j_{p})} \sum_{q} B_{qk}(j) x_{q}^{(j_{q})} \right) dt
\]

and equation (4) can be rewritten as

\[
\frac{dD}{dt} = \sum_{p} \det Q^{(p)} + \frac{1}{2} \sum_{p,q,j} \det R^{(p,q,j)} \, dt,
\]

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where $Q^{(p)}$ are matrices of the type $n \times n$ defined by

$$Q_{ij}^{(p)} = x_i^{(j)} \text{ if } i \neq p \text{ and } Q_{pj}^{(p)} = dx_p^{(j)},$$

$R^{(p,q,j)}$ are matrices of the type $n \times n$ defined by

$$R_{u,v}^{(p,q,j)} = x_u^{(v)} \text{ if } u \neq p \text{ and } u \neq q,$$

$$R_{p,v}^{(p,q,j)} = \sum_k B_{pk}^{(j)} x_k^{(v)}, \quad R_{q,v}^{(p,q,j)} = \sum_k B_{qk}^{(j)} x_k^{(v)}.$$

Equation (5) can be easily transformed (by using well-known properties of determinants) into

$$dD = D(t) \left( \sum_p A_{pp} dt + \sum_{p,j} B_{pj}^{(j)} dW_j \right) +$$

$$+ \frac{1}{2} \sum_{p,q,j} \left( B_{pp}^{(j)} B_{pq}^{(j)} - B_{pq}^{(j)} B_{pp}^{(j)} \right) D(t) dt$$

which is the same equation as (2).

**Conclusion 1.** Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_0) = I$ (I is the unit matrix) then

$$\det X(t) = D(t) = \exp \left\{ - \int_{t_0}^t \operatorname{tr} A(\tau) d\tau \right\} -$$

$$- \frac{1}{2} \sum_j \int_{t_0}^t \operatorname{tr}(B^{(j)}(\tau))^2 d\tau + \sum_j \int_{t_0}^t \operatorname{tr} B^{(j)}(\tau) dW_j(\tau) \right\}.$$

The formula for $D(t)$ follows immediately from (2) and the Itô formula.

**Conclusion 2.** Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_0) = I$ then the probability that $X(t)$ is regular for all $t \in (t_0, \infty)$ is equal to one.
This conclusion follows directly from formula (6). Conclusion 2 implies that the inverse matrix $X^{-1}(t)$ exists almost everywhere.

**Conclusion 3.** Let the assumptions of Theorem 1 be fulfilled. If $X(t)$ is the fundamental matrix solution of (1), $X(t_0) = I$ then to every $T > t_0$ and $\alpha \geq 1$ there exists $C \geq 0$ such that $E \|X^{-1}(t)\|^\alpha \leq C$ for $t \in (0, T)$.

**Proof.** If $X^{-1}(t)$ exists then $X^{-1}(t) = (-1)^{k+\ell} \det X(\ell, k) / \det X$ where $X(\ell, k)$ is the submatrix of $X$ corresponding to the element $x_{\ell, k}$. Since $\det X(\ell, k) = \sum e (j_1, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_n) \prod_{s \in \ell} x_{s}$ where $j_1, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_n$ are permutations of $1, 2, \ldots, k-1, k+1, \ldots, n$ we can derive an estimate

\[
E \left| \frac{\det X(\ell, k)}{\det X} \right|^\alpha \leq ((n - 1)!)^\alpha \sum_{j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_n} \prod_{s \in \ell} x_{s} \left[ \prod_{s \in \ell} x_{s} \right]^\alpha \frac{\prod_{s \in \ell} \sqrt{E} x_{s}^{\alpha \epsilon n}}{\det X} \left( \prod_{s \in \ell} x_{s} \right)^{\alpha \epsilon n}
\]

where $E$ is the mathematical expectation. It is proved in [2] that to every $T > t_0$, $\alpha \geq 1$ there exists $C \geq 0$ such that $E \|x(t)\|^\alpha n \leq C$ for $t \in (t_0, T)$ where $x(t)$ is a solution of (1) fulfilling $\|x(t_0)\| \leq 1$. Using (6) we obtain that also $E \frac{1}{|\det X|^\alpha n} \leq C$ for $t \in (0, T)$. Inequality (7) implies $E \frac{\det X(\ell, k)}{\det X} \leq ((n - 1)!)^\alpha C \frac{n-1}{n\alpha n}$ and the statement of
Conclusion 3 easily follows.

**Theorem 2.** Let \( A(t), B^{(j)}(t), w_j(t), j = 1, \ldots, k \) fulfill the conditions of Theorem 1 and let \( \alpha(t), \beta_j(t), j = 1, \ldots, k \) be \( n \)-dimensional vector functions defined on \( (0, \infty) \) such that \( \| \alpha(t) \|, \| \beta_j(t) \| \) are locally integrable. Denote by \( X(t) \) the fundamental matrix solution of (1), \( X(t_0) = I \). If \( x_0 \) is a nonstochastic vector then the process

\[
x(t) = X(t)x_0 + X(t) \int_{t_0}^{t} X^{-1}(\tau)(\alpha(\tau) - \sum_j B^{(j)}(\tau) \beta_j(\tau))d\tau + X(t) \int_{t_0}^{t} X^{-1}(\tau) \sum_j \beta_j(\tau)dw_j(\tau)
\]

is the solution of the nonhomogeneous Itô equation

\[
dx = A(t)xdt + \sum_{j=1}^{k} B^{(j)}(t)x dw_j + \alpha(t)dt + \sum_{j=1}^{k} \beta_j(t)dw_j
\]

fulfilling \( x(t_0) = x_0 \).

**Proof.** With respect to Conclusion 2 the process \( X^{-1}(\tau) \) exists and the integrals converge. Denote \( J_1(t) = X(t)x_0 \), \( J_2(t) = X(t) \int_{t_0}^{t} X^{-1}(\tau)(\alpha(\tau) - \sum_j B^{(j)}(\tau) \beta_j(\tau))d\tau \) and

\[
J_3(t) = X(t) \int_{t_0}^{t} X^{-1}(\tau) \sum_j \beta_j(\tau)dw_j(\tau).
\]

The process \( J_1(t) \) is evidently the solution of (1) fulfilling \( J_1(t_0) = x_0 \). Using the Itô formula we obtain that \( J_2(t) \) is the solution of

\[
dJ_2 = AJ_2 dt + \sum_j B^{(j)}J_2 dw_j + (\alpha - \sum_j B^{(j)} \beta_j)dt
\]

fulfilling \( J_2(t_0) = 0 \) and the process \( J_3(t) \) is the solution of
\[ dJ_3 = AJ_3 \, dt + \sum_j B(j) J_3 \, dw_j + \sum_j \beta_j \, dt + \sum_j \beta_j \, dw_j \]

fulfilling \( J_3(t_0) = 0 \).

**Remark.** The theorems and the conclusions are valid even if \( A(t), B(t), \omega(t), \beta_j(t) \) are nonanticipative stochastic processes fulfilling the above conditions with probability 1.

**References**


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