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A MIXED BOUNDARY VALUE PROBLEM FOR HEAT POTENTIALS

(Preliminary communication)

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Abstract: The applicability of the method of integral equations to the mixed boundary value problem for the heat equation on the set $D \times]\tau_1, \tau_2[\subset R^{m+1}$ is investigated. No a priori smoothness restrictions on the boundary of the open set $D \subset R^m$ are assumed. A weak characterization of the boundary condition is introduced.

Key words: Heat equation, heat potentials, BVP, Fredholm method.

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Suppose that D is an arbitrary open subset of R^m $(m \geq 1)$ having compact and non-void boundary ∂D, $\tau_1 < \tau_2$ are real numbers, $E = D \times]\tau_1, \tau_2[$ and $C = \partial D \times [\tau_1, \tau_2[$. We denote by $\mathcal{L}$ the Banach space of all finite signed Borel measures μ such that $|\mu| (R^{m+1} \setminus C) = 0$. In what follows, λ will be a fixed non-negative element of $\mathcal{L}$ and with each $\mu \in \mathcal{L}$ we associate the potentials

$$\mathcal{U}\mu(z) = \int G(z - \zeta) d\mu(\zeta), \quad \mathcal{U}^{*}\mu(z) = \int G(\zeta - z) d\mu(\zeta).$$

Here G denotes the well known kernel

$$G(z) = \begin{cases} z_{m+1}^{-m/2} \exp\left(-\sum_{j=1}^{m} z_j^2 / 4z_{m+1}\right), & z_{m+1} > 0 \\ 0 & , \quad z_{m+1} \leq 0 \end{cases}$$

connected with the heat equation.
Remark that $\mathcal{U}\mu$ satisfies the heat equation on the complement of spt μ.

Given $\mu \in \mathcal{L}$, let $\mathcal{D}_{\tau_2}(\mu)$ denote the set of all infinitely differentiable functions φ with compact support contained in $R^m \times]-\infty, \tau_2[$ such that $\int |\varphi|\, |\mathcal{U}|\mu|\, d\lambda < \infty$. Since $\mathcal{U}\mu$ and the partial derivatives $\partial_j \mathcal{U}\mu$ $(j = 1,\dots,m)$ are locally integrable it is meaningful to define the functional $T\mu$ on $\mathcal{D}_{\tau_2}(\mu)$ as follows:

$$\langle \varphi, T\mu \rangle = \int_E [\sum_{j=1}^{m} \partial_j \mathcal{U}\mu(z)\, \partial_j \varphi(z) -$$
$$- \mathcal{U}\mu(z)\, \partial_{m+1} \varphi(z)]\, dz + \int \varphi\, \mathcal{U}\mu\, d\lambda .$$

Consider for a moment a special case. Suppose that ∂D is a smooth hypersurface, $\partial_j \mathcal{U}\mu$ are uniformly continuous on E and λ is absolutely continuous with respect to the area measure σ on C. Then the divergence theorem and the integration by parts (with respect to the time variable) yield

$$\langle \varphi, T\mu \rangle = \int_C \varphi \cdot [\frac{\partial \mathcal{U}\mu}{\partial n} + \frac{d\lambda}{d\sigma}\, \mathcal{U}\mu]\, d\sigma$$

where n denotes the outer normal to E.

We see that the functional $T\mu$ represents a weak characterization of the mixed boundary value condition.

For smooth boundaries and classical boundary conditions, the boundary value problems for the heat equation were studied by potential-theoretic methods since the very beginning of the century (for references see e.g. [3]). The same approach, however, is no longer applicable for general domains. The case of the second Fourier problem on non-regular regions (which corresponds to $\lambda = 0$) was investigated in [1]. The mixed boundary value problem for the Laplace equation with a weakly characterized boundary condition was treated in [2] (see also preceding papers published in Czechoslovak Math. J. 22(1972), 312-324, 462-489).

Now the following problem arises: Under what circumstances is $T\mu$ representable by means of an element of $\mathcal{L}$? This means: When there is a $\nu_\mu \in \mathcal{L}$ such that $\langle \varphi, T\mu \rangle = \langle \varphi, \nu_\mu \rangle$ whenever $\varphi \in \mathcal{D}_{\tau_2}(\mu)$? To give the answer, we shall recall the following notion introduced by J. Král in 1966 (cf. [1]). If $S \subset R^m$ is a segment, then $x \in S$ is termed a hit of S on D provided each neighborhood of x meets both $S \cap D$, $S \setminus D$ in a set of positive linear measure. Given $x \in R^m$, $r > 0$ and $\theta \in \Gamma = \{z;\ |z| = 1\}$, denote by $n_r(x, \theta)$ the number of all hits of $\{x + \varrho\theta\ ;\ 0 < \varrho < r\}$ on D and put

$$v_r(x) = [\sigma(\Gamma)]^{-1} \int_\Gamma n_r(x, \theta)\, d\sigma(\theta),$$

$$k_D = \lim_{r \to 0+} \sup_{x \in \partial D} v_r(x).$$

<u>Theorem 1</u>. For every $\mu \in \mathcal{L}$, $T\mu$ is representable by means of a unique element of $\mathcal{L}$ if and only if $k_D < \infty$ and the potential $\mathcal{U}^{\#}\lambda$ is bounded on C.

If this condition holds, then $T\mu$ can be identified with the representing measure and $T: \mu \longmapsto T\mu$ appears to be a bounded linear operator on $\mathcal{L}$ and we may formulate the mixed boundary value problem as follows: Given $\nu \in \mathscr{L}$, find a $\mu \in \mathscr{L}$ with $T\mu = \nu$.

Let I stand for the identity operator on $\mathscr{L}$ and α be a real number. Denote by ω_α the distance of the operator $T - \alpha I$ from the subspace of compact operators acting on $\mathscr{L}$ and put

$$a_T = \inf_{\alpha \neq 0} (\omega_\alpha / |\alpha|).$$

Note that if $a_T < 1$, then the Riesz-Schauder theory is applicable to the equation $T\mu = \nu$ on $\mathscr{L}$. Using some results of [1], we have the following

Theorem 2. Let $\partial D = \partial(R^m \setminus \bar{D})$ and $\mathcal{U}^{*}\lambda$ be continuous on R^{m+1}. Then $a_T < 1$ if and only if $k_D < \frac{1}{2}$.

The important information about the kernel of T is contained in the following theorem whose proof is rather complicated.

Theorem 3. Suppose that

(i) $\partial D = \partial(R^{m+1} \setminus \bar{D})$ and $k_D < \frac{1}{2}$,

(ii) $\mathcal{U}\lambda$ and $\mathcal{U}^{*}\lambda$ are continuous on R^{m+1}.

If $\mu \in \mathscr{L}$ and $T\mu = 0$, then the potential $\mathcal{U}\mu$ is uniformly continuous on $(R^m \times]\tau_1, \tau_2[) \setminus C$.

Corollary. The operator T is one-to-one on $\mathscr{L}$.

Combining this result with Theorem 2 we have the first part of the following

Theorem 4. Assume (i) and (ii). Then for every $\nu \in \mathscr{L}$ there is a uniquely determined $\mu \in \mathscr{L}$ such that $T\mu = \nu$. If, moreover, λ and ν have a density with respect to the area measure on C, the same is true for μ .

The above results were presented on the Czechoslovak conference on differential equations and their applications (Equadiff 4, Prague, August 1977). The proofs of the announced theorems together with further related results and details and the corresponding bibliography are contained in a thesis submitted to the Faculty of Mathematics and Physics (Charles University, Prague) and will be later published elsewhere.

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