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FACTORIZATION IN THE ALGEBRA OF RAPIDLY DECREASING FUNCTIONS ON $\mathbb{R}^n$

Hana PETZELTOVÁ and Pavla VHOVÁ, Praha

Abstract: A factorization theorem which is an analogy of factorization theorems in Banach algebras is proved in the algebra of rapidly decreasing functions on $\mathbb{R}^n$. The result is closely related to investigations of existence of factorization in Fréchet algebras with an approximate unit.

Key words: Rapidly decreasing function, approximate unit, Fréchet algebra.

AMS: 46E25

Let $\mathbb{R}^n$ be n-dimensional Euclidean space. As usual, denote by $|t| = (t_1^2 + \ldots + t_n^2)^{1/2}$ for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and $D^k_x = \frac{\partial^{|k|}}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}$ for $x \in C^\infty(\mathbb{R}^n)$, $k = (k_1, \ldots, k_n) \geq 0$,

$|k| = k_1 + \ldots + k_n$. Let us recall that $i = (i_1, \ldots, i_n) \preceq k = (k_1, \ldots, k_n)$ by definition if $i_1 \leq k_1$, $i_2 \leq k_2$, $\ldots$, $i_n \leq k_n$ and $0 = (0, 0, \ldots, 0)$. As usual, $\binom{k}{i} = \binom{k_1}{i_1} \ldots \binom{k_n}{i_n}$.

We shall denote by $\mathcal{Y}$ the subalgebra of $C^\infty(\mathbb{R}^n)$ consisting of all functions rapidly decreasing at infinity, i.e.

$\mathcal{Y} = \left\{ x \in C^\infty(\mathbb{R}^n) : \sup_{t \in \mathbb{R}^n} |t|^j |D^k x(t)| < \infty \text{ for all non-negative integers } j \text{ and multiindices } k \right\}$.
with the topology generated by the system of pseudonorms

\[ |x|_{jk} = \max_{t \in R_n} \sum_{i=1}^n t_i^j |(D^k x)(t)|. \]

Concerning the problem of factorization in projective limits of Banach algebras there exists an approximate unit in the algebra \( \mathcal{G} \) which may be regarded as the projective limit of Banach algebras \( \mathcal{G}_{jk} \) consisting of all functions from \( C^\infty(R_n) \) for which the norm \( \max_{0 \leq i, k} |j_1| \) is finite. Namely, the system of characteristic functions of \( D_k = (t \in R_n, |t| \leq k) \) \( (k = 1, 2, \ldots) \) in \( R_n \) smoothened by convolution with suitable functions from \( C^\infty(R_n) \) forms an approximate unit, unfortunately, this unit is unbounded in each \( \mathcal{G}_{jk} \). It turns out that the iterative process which often provides a positive solution in many proofs of factorization theorems (see, for example, [1] - [9]) fails to converge here. Nevertheless, it is possible to prove existence of power factorization on bounded subsets of \( \mathcal{G} \) with the help of special properties of the algebra \( \mathcal{G} \).

1. Preliminaries. Denote by \( w \) the function \( w(t) = |t| \) for \( 0 \neq t \in R_n \). The function \( w \) is of class \( C^\infty \). Since

\[
\frac{\partial w^p}{\partial t_s} (t) = p \cdot w^{p-2}(t) \cdot t_s \quad \text{for every integer } p,
\]

it follows by induction that

\[
(D^k w^p)(t) = \sum_{0 \leq i, k} c(i, k, p) |t|^{|p-k|-1} t_i^1.
\]
If $e^s = (\sigma^s_1, \ldots, \sigma^s_n)$ ($s = 1, 2, \ldots, n$) then $(D^{k+e^s}w^p)(t) =$

$= \sum_{0 \leq l \leq k} c(1,k,p)(p-|k|-\|l\|)|t|^{p-|k|-\|l\|}t^l e^s + \sum_{0 \leq l \leq k} c(1,k,p) |t|^{p-|k|-\|l\|} t^l e^s$. Hence, if $p > 0$,

$|c(1,k + e^s,p)| \leq \max (2p,3|k|) \cdot \max |c(r,k,p)|$. The last inequality can be easily proved by considering all possible cases. This, together with $c(0,0,p) = 1$, yields

$|c(1,k,p)| \leq (\max (2p,3|k|)) |k|$

$(D^k w^p)(t) \leq d_k p |k| |t|^{p-|k|}$ for $t \neq 0$, $k \geq 0$, $p > 0$.

1.1. Lemma. There exist positive constants $C, C_k$ ($k \geq 0$) such that, for every sequence $(m(p))_{p=1}^{\infty}$ of natural numbers with $m(p + 1) - m(p) \geq 5$ for $p = 1, 2, \ldots$ and $m(1) \geq 3$, there exists a positive function $b \in C^\infty(R_+)$ satisfying:

$1^0$ $b(t) = 1$ for $|t| < m(1) - 2$

$2^0$ $|D^k b(t)| \leq C_k (m(p) + 2)^P$

$3^0$ $b(t) \geq C \cdot (m(p) - 2)^{P-1}$

for $|t| \leq m(p) - 2, m(p) + 2$,$p = 1, 2, \ldots$, $k \geq 0$.

Proof. Let $(m(p))_{p=1}^{\infty}$ be a sequence of natural numbers satisfying $m(p + 1) - m(p) \geq 5$ ($p = 1, 2, \ldots$) and $m(1) \geq 3$. We shall construct a function $b$ having the required properties and we shall show that the corresponding constants $C_k, C$ do not depend on the choice of $(m(p))_{p=1}^{\infty}$. Let $a$ be a positive function defined by

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\[ a(t) = \begin{cases} 
1 & \text{for } |t| \leq m(1) \\
|t|^p & \text{for } |t| \geq (m(p), m(p + 1)) 
\end{cases} \quad p = 1, 2, \ldots \]

We shall modify the function \( a \) so as to obtain a \( C^\infty \) function. Take a function \( \varphi \in C^\infty (\mathbb{R}_0) \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \) in a neighbourhood of zero, \( \text{supp} \varphi \) is equal to the unit disc \( D_1 \) in \( \mathbb{R}_n \), \( \varphi \) positive inside \( D_1 \) and \( D^k \varphi = 0 \) on \( \partial D_1 \) for all nonnegative multiindices \( k \). Denote by \( N_k = \max |(D^k \varphi)(t)| \quad (k \geq 0) \). Let \( \varphi_p \) be the function defined as follows:

\[ \varphi_p(t) = \begin{cases} 
\varphi(t - m(p) \frac{t}{|t|}) & \text{for } t \neq 0 \\
0 & \text{for } t = 0 
\end{cases} \]

Clearly, the function \( \varphi_p \) is a well defined function of class \( C^\infty \) and \( \text{supp} \varphi_p = \{ t : m(p) - 1 \leq |t| \leq m(p) + 1 \} \). We shall show that, for every \( k \geq 0 \), there exist constants \( K_k \) (depending on \( k \) only) such that \( \sup_{t \in \mathbb{R}_n} |(D^k \varphi_p)(t)| \leq K_k \) for all \( k \geq 0 \) and \( p = 1, 2, \ldots \). Denote \( \varphi^p_1(t) = t_i (1 - m(p)|t|^{-1}) \) for \( |t| \leq (m(p) - 1, m(p) + 1) \). Since

\[ (D^s \varphi^p_1)(t) = \sum_{i=1}^n (D^s \varphi)(t(l - m(p)|t|^{-1}))(D^s \varphi^p_1)(t) \]

it follows by induction that \( (D^k \varphi^p_1)(t) \) is a polynomial of order \( |k| + 1 \) in indeterminates \( (D^j \varphi)(t(l - m(p)|t|^{-1}))(D^j \varphi^p_1)(t) \) (\( 1 \leq j \leq |k| \), \( D^j \varphi^p_1)(t) \) \( 0 \leq i \leq k, i = 1, 2, \ldots, n \)). Hence, it is sufficient to show that the derivatives of \( \varphi^p_1 \) are bounded by constants which do not depend on \( p \). We have

\[ (D^j \varphi^p_1)(t) = \sigma_{ij} - \sigma_{ij} t_i (D^j \varphi)(t) + m(p) t_i t_j |t|^{-3} \quad \text{for } j = e^s \]

and

\[ (D^j \varphi^p_1)(t) = -m(p) [t_i (D^j \varphi)(t) + j_1 (D^j - e^i \varphi)(t)] \quad \text{for } |j| \geq 2. \]
According to (1) there exist constants $\varepsilon_j$ such that, for $|j| \geq 1$, $|t| \in (m(p) - 1, m(p) + 1)$,

$$|\langle D_j \varphi_{\lambda} \rangle(t) - \varepsilon_j m(p)| |t|^{-|j|} \leq 2\varepsilon_j.$$  

Now, set

$$b(t) = \begin{cases} 
\langle (\gamma_p \sigma \varphi_t ) + (1 - \varphi_p(t))a(t) \rangle & \text{if } |t| \leq (m(p) - 2, m(p) + 2), \\
a(t) & \text{otherwise}.
\end{cases}$$

This function belongs to $C^\infty(R^n)$ and satisfies 1°. Given $a |t| \in (m(p) - 2, m(p) + 2)$, $t \notin m(p)$ we have, according to (1), (2),

$$|D^k b(t)| \leq \int \varphi_p(x) a(x) (D^k \varphi)(t-x) dx + \sum_{i \in \mathbb{Z}^k} \left( k \right)_i |(D^i (1 - \varphi_p))(t)(D^{k-i} a(t))| \leq N_k \sup_{|x| \leq (m(p) - 1, m(p) + 1)} a(x) \mathcal{H}_n \{x: |t-x| \leq 1\} + \sum_{i \in \mathbb{Z}^k} \left( k \right)_i K_i \sum_{0 \leq j \leq k-i} \max(2p, 3|k-i|)|k-i|.$$ 

$$|t|^{p-|k-i|-|j|} \leq N_k \mathcal{H}_n \{x: |t-x| \leq 1\} |(m(p) + 1)p + \sum_{i \in \mathbb{Z}^k} \left( k \right)_i K_i \sum_{0 \leq j \leq k-i} \max(2p, 3|k-i|)|k-i| \cdot |t|^{-|k-i| (m(p) + 1)p + 2p \leq m(p) - 2, p = 2, 3, \ldots}
$$

To obtain estimate (4) denote by $M = \{t; \varphi(t) \leq 1/2\}$ and $M_p = \{t; \varphi_p(t) \leq 1/2\}$. Then

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\[b(t) \geq \frac{1}{2} a(t) \text{ for } t \in M_p \text{ and so } b(t) \geq \frac{1}{2} (m(p-2))^{P-1}\]

for \( t \in \{t \mid t < (m(p)-2, m(p)+2) \} \setminus M_p \). Observe that \( M_p = \{ t \mid t = m(p)t/t|t| \in M \} \). Take \( \varepsilon, \varepsilon' \) positive such that \( t; |t| < \varepsilon \} \subset M \setminus \{ t; |t| < \varepsilon' \} \) and \( \varepsilon + \varepsilon' < 1 \). If \( 1 > \varepsilon > \varepsilon + \varepsilon' \) then, for each \( t \in M_p \), the set \( K_p = \{ x \in M : |x-t| < \varepsilon \} \)
contains a ball \( K'_p = \{ x \in \mathbb{R}^n : |x-m(p)t|t| < \varepsilon \} \) for all \( p = 1, 2, \ldots \). Then, for \( t \in M_p \), we have

\[b(t) \geq \int_{K_p} a(x) \varphi(t-x) dx \geq \inf_{x \in K_p} a(x). \inf_{z \in K'_p} \varphi(z).\]

where \( \varphi \) is a constant depending on \( \varepsilon, \varepsilon, n, \varphi \) only.

We have to show now that \( b^{-1} \in \mathcal{G} \), i.e.

\[\sup_{p} \max_{t \in (m(p)-2, m(p)+2)} |t|^{j} \langle D^{k}b^{-1}(t) \rangle < \infty \text{ for all } j, k \geq 0.\]

If \( f, f^{-1} \in C_0^{\infty}(\mathbb{R}^n) \) then \( D^{k}f^{-1} = P_k(f, \ldots, D^{k}f)/f^{(|k|+1)} \)

where \( P_k \) is a polynomial of order \( |k| \) in indeterminates \( D^i f \) for \( 0 \leq i \leq k \) and the coefficients of \( P_k \) depend on \( k \) only.

Fix \( j \) and \( k \). First, let us take \( |t| < (m(p)+2, m(p+1)-2) \) for \( 2p \geq |k| \). Then \( b(t) = w^p(t) = |t|^p \). According to (1), (2), and (5) we have, for arbitrary nonnegative multiindex \( m \leq k \)

\[|D^{m}w^p(t)| \leq d_m |t|^m |t|^p-|m| \leq d_m |t|^p\]

It follows that

\[|t|^{j} \langle D^{k}b^{-1}(t) \rangle = |t|^{j} \langle D^{k}w^{-p}(t) \rangle = |t|^{j} \langle P_k(w^p(t), \ldots, (D^{k}w^p)(t) \rangle \cdot w^{-p}(|k|+1) \cdot \max \{ |D^{j}w^p(t)\} \cdot |k| \cdot t|^{-p}(|k|+1) \cdot \bar{M}_k \cdot |t|^{-j-p} \cdot \bar{M}_k \cdot |t|^{-j-p} \]

where \( \bar{M}_k, M'_k \) are suitable constants.

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Now assume $|t| < m(p)^{-2}f^{ia(p)}$. We have, according to 2°, the following estimate

$$|t|^j ((p^k b^{-1})(t)| \leq |t|^j \frac{C_k'(m(p)+2)^p}{C_k|kl+1| (m(p)-2)(p-1)(l|kl+1|) \leq C_k''(1+\frac{4}{m(p)-2})^{p|kl+1|} \cdot (m(p)-2)^{-p+j+|kl+1|} \leq C_k''(1+\frac{4}{p})^{p|kl+1|} \cdot (m(p)-2)^{-p+j+|kl+1|}.$$ 

The last expression is bounded and so the proof is complete.

2.1. Theorem. Let $K$ be a bounded subset of $\mathcal{G}$. Given $\varepsilon > 0$, $s_0$ natural and $j_0 \geq 0$, $k_0 = (k_1,\ldots,k_m) \geq 0$, there exists an $a$ in $\mathcal{G}$ and a sequence $(K_s)_{s=1}^\infty$ of bounded subsets of $\mathcal{G}$ with the following property: for $x \in K$ there exists an $y_s \in K_s \cap (\mathcal{G}x)^-$ with

1° $x = a^s y_s$ for $s = 1,2,\ldots$
2° $|x-y_s|_{j_0 k_0} \leq \varepsilon$ for $s = 1,2,\ldots,s_0$.

Moreover, if $x_n \to 0$ in $\mathcal{G}$ then the corresponding $y_s$ in $\mathcal{G}$ for each $s$.

Proof. The subset $K$ is bounded, so there are a positive function $h$ and nonnegative constants $(q_k)_{k \geq 0}$ such that

$$\sup_{t \in R_n} |t|^j h(t) \leq M_j < \infty \quad \text{for all } j \geq 0 \quad \text{and} \quad |(D^k x)(t)| \leq q_k h(t)$$

for all $t \in R_n$, $k \geq 0$, $x \in K$ (see [10], p. 235). Denote by $Q_p = \max q_k$ and take a sequence $(\varepsilon_p)_{p=1}^\infty$ of positive numbers. Since $p^{k w^p}$ is a polynomial in indeterminates $|t|$, $|t|^{-1}, t_1,\ldots,t_n$ ($t \not= 0$) it follows from (2) that we can find a sequence $(m(p))_{p=1}^\infty$ such that
(i) \[ |t|^j |(D^k w^r)(t)| h(t) \leq \varepsilon_p \] for \[ |t| \geq m(p)-2 \] and all \[ |k| \leq p, \ 0 \leq r \leq p^2, \ j \leq p(p+1) \]

(ii) \[ |t|^j \epsilon_0^r \sigma |(D^k w^r)(t)| h(t) \leq \varepsilon_p \] for all \[ p = 1, 2, \ldots, \]
\[ 0 \leq r \leq s_0 p, \ 0 \leq k \leq k_0, \ |t| \geq m(p)-2 \]

(iii) \[ m(p+1)-m(p) \geq 5, \ m(1) \geq 3 \]

It follows that

\[ |t|^j |(D^k w^r)(t)| |(D^l x)(t)| \leq Q_p \varepsilon_p \]
for \[ |t| \geq m(p)-2, \ |l| \leq p, \ 0 \leq r \leq p^2, \ j \leq p(p+1), \ x \in K \]
and

\[ |t|^j \epsilon_0^r \sigma |(D^k w^r)(t)| |(D^l x)(t)| \leq Q_k \varepsilon_p \]
for \[ |t| \geq m(p)-2, \ 0 \leq i, k \leq k_0, \ 0 \leq r \leq s_0 p, \ x \in K \]
and \[ p = 1, 2, \ldots \]

Let us take an \[ x \in K \] and consider the factorization of \[ x \] in the form \[ x = b^{-s}(b^{s}x) \] where \[ b \] is the function corresponding to \[ (m(p))_{p=1}^\infty \] according to Lemma 1.1. The function \[ b^{-1} \] belongs to \[ \mathcal{Y} \] so that \[ b^{-s} \] belongs to \[ \mathcal{Y} \] as well. We have to show that \[ b^{s}x \] is in \[ \mathcal{Y} \] (\[ s = 1, 2, \ldots \]), i.e.

\[ \sup_{p \geq 1} \left\{ \max_{m(p)-2 \leq |t| \leq m(p+1)-2} \right\} \]

\[ \max_{m(p)-2 \leq |t| \leq m(p)+2} \] \[ |t|^j |(D^k b^{s}x)(t)| < \infty \] for all \[ j \geq 0, \]
\[ s \geq 1, \ k \geq 0. \]

Having known that \[ b^{s}x \in \mathcal{Y} \] it follows that \[ b^{s}x \in (\mathcal{Y} x)^{-}. \] Indeed, there exists an approximate unit \[ (e_{m})_{m=1}^\infty \] in \[ \mathcal{Y} \] consisting of functions with compact supports, so that \[ b^{s}x = \lim e_{n} b^{s}x \] for \[ x \in \mathcal{Y}, \ s = 1, 2, \ldots. \] Since \[ e_{n} b^{s}x \] are in \[ \mathcal{Y} \] as functions with compact supports, we obtain \[ b^{s}x \in (\mathcal{Y} x)^{-}. \]

Fix \( j, k \) and \( s. \) If \[ p > \max(|k|, j, s) \] we obtain according to (6), for \[ |t| \leq m(p)+2, m(p)+2 \geq 2 \), the estimate

\[ |t|^j |(D^k (b^{s}x))(t)| = |t|^k |(D^k (b^{s}x))(t)| = \]

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Now, assume $|t| \in (m(p)-2, m(p)+2)$. It is easy to see that, for $f \in C^\infty(R_n)$, $D^k f$ is a polynomial $P_{k,s}$ of order $s$ in indeterminates $f, \ldots, D^k f$. Again, according to (6), $|t| p^2 + p$.

$|(D^k f)(t)| \leq Q_{p} p$. This, together with $2^o$ of 1.1 yields

$|t|^j |(D^k b^s f)(t)| \leq |t|^j \sum_{i=0}^{k} |(D^i b^s f)(t)| |(D^k f)(t)|$

$= \sum_{i=0}^{k} |P_{i,s}(b(t), \ldots, (D^i b)(t))| |t|^j |(D^k f)(t)|$

$\leq \sum_{i=0}^{k} K_{i,s} (m(p)+2)^s P_{p^2} |t|^p - p^2 |t|^p |(D^k f)(t)|$

$\leq (m(p)+2)^s P_{p^2} (m(p)-2)^-p^2 Q_p p$.

Given an $\varepsilon > 0$, let us choose now $Q_{p} p = \varepsilon/2$. Using these estimates we can deduce the following facts. First, $b^s f$ for $f \in F^s$ for all $b^s K = K$ are bounded in $F$ and $|x-b^s f|_{\infty} = \max |t| ^{j_0}$

$|D^* f(t)-D^* b^s f(t)| = \max |t|^{j_0} |D^* f(t)-D^* b^s f(t)|$

$\leq \varepsilon$ for $s=1,2, \ldots, s_0$. Finally, if $x_n \in F$ then, for $K = (x_n)^{s_0}$, we obtain $y_n = b^s x_n$ tends to zero as well. Indeed, let us fix $j, k$ and $s$. Given an $\varepsilon > 0$, let us find
so that $M_q^p \in p \leq \mathcal{E}$ for $p \geq p_0$. There exists $n_0$ such that, for $n \geq n_0$, 
$$|x_n|_1 \leq \left( \sum_{|\alpha| \leq 1} \left( \sup_{t \in \mathbb{R}} |(D^{\alpha}b^s)(t)| \right)^{-1} \right) \sup_{|t| \leq m(p_0)} |(D^i b^s)(t)|$$
for $1 \leq k$. It follows that, for $n \geq n_0$, we have
$$|y_n|_k = \max_{|t| \leq m(p_0)} |t|^j |(D^k b^s x_n)(t)| + \max_{|t| \leq m(p_0)} |t|^j |(D^k b^s x_n)(t)| \leq \varepsilon$$

The proof is complete.

Remark. Since the Fourier transformation is a continuous linear mapping of $\mathcal{F}$ onto itself and takes the pointwise multiplication to the convolution, Theorem 2.1 holds also if we replace the multiplication by the convolution.

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