Commentationes Mathematicae Universitatis Carolinae

Hana Petzeltová; Pavla Vrbová

Factorization in the algebra of rapidly decreasing functions on $\mathbf{R}_n$

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 3, 489--499

Persistent URL: http://dml.cz/dmlcz/105871

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

19,3 (1978)

FACTORIZATION IN THE ALGEBRA OF RAPIDLY DECREASING FUNCTIONS ON R_n

Hana PETZELTOVÁ and Pavla VRBOVÁ, Praha

Abstract: A factorization theorem which is an analogy of factorization theorems in Banach algebras is proved in the algebra of rapidly decreasing functions on R_n. The result is closely related to investigations of existence of factorizaion in Fréchet algebras with an approximate unit.

Key words: Rapidly decreasing function, approximate unit, Fréchet algebra.

AMS: 46E25

Let R_n be n-dimensional Euclidean space. As usual, denote by $|t| = (t_1^2 +\dots+ t_n^2)^{1/2}$ for $t = (t_1,\dots,t_n) \in R_n$ and

$$D^k x = \frac{\partial^{|k|} x}{\partial^{k_1} t_1 \dots \partial^{k_n} t_n} \quad \text{for } x \in C^\infty(R_n),\ k = (k_1,\dots,k_n) \geq 0,$$

$|k| = k_1 +\dots+ k_n$. Let us recall that $i = (i_1,\dots,i_n) \leq k = (k_1,\dots,k_n)$ by definition if $i_1 \leq k_1$, $i_2 \leq k_2, \dots, i_n \leq k_n$ and $0 = (0,0,\dots,0)$. As usual, $\binom{k}{i} = \binom{k_1}{i_1} \dots \binom{k_n}{i_n}$.

We shall denote by $\mathcal{S}$ the subalgebra of $C^\infty(R_n)$ consisting of all functions rapidly decreasing at infinity, i.e.

$$\mathcal{S} = \left\{ x \in C^\infty(R_n) : \sup_{t \in R_n} |t|^j |(D^k x)(t)| < \infty \text{ for all non-negative integers } j \text{ and multiindices } k \right\} =$$

$$= \left\{ x \in C^{\infty}(R_n) : \sup_{t \in R_n} |t^i (D^k x)(t)| < \infty \text{ for all multi-indices } i,k \geq 0 \quad (t^i = t_1^{i_1} \dots t_n^{i_n}) \right\}$$

with the topology generated by the system of pseudonorms $|x|_{jk} = \max_{t \in R_n} |t|^j |(D^k x)(t)|$.

Concerning the problem of factorization in projective limits of Banach algebras there exists an approximate unit in the algebra $\mathcal{S}$ which may be regarded as the projective limit of Banach algebras $\mathcal{S}_{jk}$ consisting of all functions from $C^{\infty}(R_n)$ for which the norm $\max_{0 \leq i \leq k} | \ |_{ji}$ is finite. Namely, the system of characteristic functions of $D_k = (t \in R_n, |t| \leq k)$ $(k = 1,2,\dots)$ in R_n smoothened by convolution with suitable functions from $C^{\infty}(R_n)$ forms an approximate unit, unfortunately, this unit is unbounded in each $\mathcal{S}_{jk}$. It turns out that the iterative process which often provides a positive solution in many proofs of factorization theorems (see, for example, [1] - [9]) fails to converge here. Nevertheless, it is possible to prove existence of power factorization on bounded subsets of $\mathcal{S}$ with the help of special properties of the algebra $\mathcal{S}$.

1. <u>Preliminaries</u>. Denote by w the function $w(t) = |t|$ for $0 \neq t \in R_n$. The function w is of class C^{∞}. Since $\frac{\partial w^p}{\partial t_s}(t) = p \cdot w^{p-2}(t) \cdot t_s$ for every integer p, it follows by induction that

(1) $$(D^k w^p)(t) = \sum_{0 \leq l \leq k} c(l,k,p) |t|^{p-|k|-|l|} t^l .$$

If $e^s = (\delta_{1s}, \ldots, \delta_{ns})$ $(s = 1,2,\ldots,n)$ then $(D^{k+e^s}w^p)(t) =$

$$= \sum_{0 \le l \le k} c(l,k,p)(p - |k| - |l|)|t|^{p-|k|-|l|-2} t^{l+e^s} +$$

$$+ \sum_{e^s \le l \le k} c(l,k,p) l_s . |t|^{p-|k|-|l|} t^{l-e^s}.$$ Hence, if $p > 0$,

$|c(l,k + e^s,p)| \le \max(2p, 3|k|) . \max_{0 \le r \le k} |c(r,k,p)|$. The last inequality can be easily proved by considering all possible cases. This, together with $c(0,0,p) = 1$, yields

$$(2) \quad \begin{aligned} &|c(l,k,p)| \le (\max(2p, 3|k|))^{|k|} \\ &|(D^k w^p)(t)| \le d_k\, p^{|k|}\, |t|^{p-|k|} \quad \text{for } t \neq 0,\ k \ge 0,\ p > 0. \end{aligned}$$

1.1. Lemma. There exist positive constants C, C_k $(k \ge 0)$ such that, for every sequence $(m(p))_{p=1}^{\infty}$ of natural numbers with $m(p + 1) - m(p) \ge 5$ for $p = 1,2,\ldots$ and $m(1) \ge 3$, there exists a positive function $b \in C^{\infty}(R_n)$ satisfying:

1^o $b(t) = 1$ for $|t| \le m(1) - 2$

$b(t) = w^p(t) = |t|^p$ for $|t| \in \langle m(p) + 2, m(p + 1) - 2 \rangle$

(3) 2^o $|(D^k b)(t)| \le C_k (m(p) + 2)^p$

(4) $b(t) \ge C \cdot (m(p) - 2)^{p-1}$

for $|t| \in \langle m(p) - 2, m(p) + 2 \rangle$, $p = 1,2,\ldots$, $k \ge 0$

3^o $b^{-1} \in \mathcal{S}$.

Proof. Let $(m(p))_{p=1}^{\infty}$ be a sequence of natural numbers satisfying $m(p + 1) - m(p) \ge 5$ $(p = 1,2,\ldots)$ and $m(1) \ge 3$. We shall construct a function b having the required properties and we shall show that the corresponding constants C_k, C do not depend on the choice of $(m(p))_{p=1}^{\infty}$. Let a be a positive function defined by

$$a(t) = \begin{cases} 1 \text{ for } |t| \le m(1) \\ |t|^p \text{ for } |t| \in \langle m(p), m(p+1)\rangle \qquad p = 1,2,\dots \end{cases}$$

We shall modify the function a so as to obtain a C^∞ function. Take a function $\varphi \in C^\infty(R_n)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ in a neighbourhood of zero, supp φ is equal to the unit disc D_1 in R_n, φ positive inside D_1 and $D^k\varphi \equiv 0$ on ∂D_1 for all nonnegative multiindices k. Denote by $N_k = \max_{t \in R_n} |(D^k\varphi)(t)|$ $(k \ge 0)$. Let φ_p be the function defined as follows

$$\varphi_p(t) = \begin{cases} \varphi(t - m(p)\dfrac{t}{|t|}) & \text{for } t \ne 0 \\ 0 & \text{for } t = 0 \end{cases}$$

Clearly, the function φ_p is a well defined function of class C^∞ and supp $\varphi_p = \{t : m(p) - 1 \le |t| \le m(p) + 1\}$. We shall show that, for every $k \ge 0$, there exist constants K_k (depending on k only) such that $\sup_{t \in R_n} |(D^k\varphi_p)(t)| \le K_k$ for all $k \ge 0$ and $p = 1,2,\dots$. Denote $\varphi_i^p(t) = t_i\,(1 - m(p)|t|^{-1})$ for $|t| \in \langle m(p) - 1, m(p) + 1\rangle$. Since $(D^{e^s}\varphi_p)(t) = \sum_{i=1}^{n} (D^{e^i}\varphi)(t(1 - m(p)|t|^{-1}))(D^{e^s}\varphi_i^p)(t)$ it follows by induction that $(D^k\varphi_p)(t)$ is a polynomial of order $|k| + 1$ in indeterminates $(D^j\varphi)(t(1 - m(p)|t|^{-1}))$ $(1 \le |j| \le |k|)$, $(D^l\varphi_i^p)(t)$ $(0 \le l \le k,\ i = 1,2,\dots,n)$. Hence, it is sufficient to show that the derivatives of φ_i^p are bounded by constants which do not depend on p. We have

$(D^j\varphi_i^p)(t) = \delta_{is} - \delta_{is}\, m(p)|t|^{-1} + m(p)t_it_s|t|^{-3}$ for $j = e^s$

and

$(D^j\varphi_i^p)(t) = -m(p)[t_i(D^jw^{-1})(t) + j_i(D^{j-e^i}w^{-1})(t)]$ for $|j| \ge 2$.

According to (1) there exist constants e_j such that, for $|j| \geq 1$, $|t| \in \langle m(p) - 1, m(p) + 1 \rangle$,

$$|(D^j \varphi_i^p)(t)| \leq e_j m(p) \; |t|^{-|j|} \leq 2 e_j .$$

Now, set

$$b(t) = \begin{cases} ((\varphi_p a) * \varphi)(t) + (1 - \varphi_p(t)) a(t) & \text{for } |t| \in \langle m(p)-2, m(p)+2 \rangle \\ & p = 1,2,\ldots \\ a(t) & \text{otherwise.} \end{cases}$$

This function belongs to $C^{\infty}(R_n)$ and satisfies 1°. Given a $|t| \in \langle m(p)-2, m(p)+2 \rangle$, $t \neq m(p)$ we have, according to (1),(2)

$$|(D^k b)(t)| \leq \left| \int \varphi_p(x) a(x) (D^k \varphi)(t-x) dx \right| + \sum_{0 \leq i \leq k} \binom{k}{i} |(D^i(1 - \varphi_p))(t) (D^{k-i} a)(t)| \leq N_k \sup_{|x| \in \langle m(p)-1, m(p)+1 \rangle} a(x) \mu_n(\{x : |t-x| \leq 1\}) + \sum_{0 \leq i \leq k} \binom{k}{i} K_i \sum_{0 \leq j \leq k-i} \max(2p, 3|k-i|)^{|k-i|} . |t|^{p-|k-i|-|j|} |t^j| \leq N_k \mu_n(\{x : |t-x| \leq 1\})(m(p)+1^p) + \sum_{0 \leq i \leq k} \binom{k}{i} K_i \sum_{0 \leq j \leq k-i} \max(2p, 3|k-i|)^{|k-i|} . |t|^{-|k-i|} (m(p) + 2)^p \leq C_k (m(p)+2)^p ,$$

where C_k are suitable constants which are clearly independent of $(m(p))_{p=1}^{\infty}$ and μ_n is the n-dimensional Lebesgue measure in R_n. The last inequality follows from

(5) $$2p \leq m(p) - 2 \qquad p = 2,3,\ldots$$

To obtain estimate (4) denote by $M = \{t ; \varphi(t) \geq 1/2\}$ and $M_p = \{t ; \varphi_p(t) \geq 1/2\}$. Then

$b(t) \geq \frac{1}{2}a(t)$ for $t \notin M_p$ and so $b(t) \geq \frac{1}{2}(m(p-2)^{p-1}$ for $t \in \{|t| \in \langle m(p)-2, m(p)+2\rangle\} \setminus M_p$. Observe that $M_p = \{t;\ t-m(p)t/|t| \in M\}$. Take $\varepsilon, \varepsilon'$ positive such that $\{t; |t| \leq \varepsilon\} \subset M \subset \{t; |t| \leq \varepsilon'\}$ and $\varepsilon + \varepsilon' < 1$. If $1 > \delta > \varepsilon + \varepsilon'$ then, for each $t \in M_p$, the set $K_p = \{x \in M_p : |x-t| < \delta\}$ contains a ball $K'_p = \{x \in R^n : |x - m(p)t|t|^{-1}| < \varepsilon\}$ for all $p = 1,2,\ldots$ Then, for $t \in M_p$, we have

$$b(t) \geq \int_{K_p} \varphi_p(x)a(x)\,\varphi(t-x)dx \geq 2^{-1} \inf_{x \in K_p} a(x) . \inf_{|z| \leq \delta} \varphi(z) .$$

$$. \mu_n(K'_p) \geq C.(m(p)-2)^{p-1}$$

where C is a constant depending on $\varepsilon, \delta, n, \varphi$ only.

We have to show now that $b^{-1} \in \mathcal{S}$, i.e.

$$\sup_p \max_{|t| \in \langle m(p)-2, m(p+1)-2\rangle} |t|^j |(D^k b^{-1})(t)| < \infty \quad \text{for all } j,k \geq 0.$$

If $f, f^{-1} \in C^\infty(R_n)$ then $D^k f^{-1} = P_k(f, \ldots, D^k f)/f^{|k|+1}$ where P_k is a polynomial of order $|k|$ in indeterminates $D^i f$ for $0 \leq i \leq k$ and the coefficients of P_k depend on k only.

Fix j and k. First, let us take $|t| \in \langle m(p)+2, m(p+1)-2\rangle$ for $2p \geq 3|k|$. Then $b(t) = w^p(t) = |t|^p$. According to (1), (2), and (5) we have, for arbitrary nonnegative multiindex $m \leq k$

$$|(D^m w^p)(t)| \leq d_m p^{|m|} |t|^{p-|m|} \leq d_m |t|^p$$

It follows that

$$|t|^j |(D^k b^{-1})(t)| = |t|^j |(D^k w^{-p})(t)| = |t|^j |P_k(w^p(t), \ldots, (D^k w^p)(t)| . w^{-p(|k|+1)}(t) \leq |t|^j M_k (\max_{0 \leq j \leq k} (|D^j w^p)(t)|)^{|k|} |t|^{-p(|k|+1)} \leq$$

$$\leq |t|^j M_k (\max_{0 \leq j \leq k} d_j)^{|k|} |t|^{p|k|} . |t|^{-p(|k|+1)} = M'_k |t|^{j-p}$$

where M_k, M'_k are suitable constants.

Now assume $|t| \in \langle m(p)-2, m(p)+2 \rangle$. We have, according to 2°, the following estimate

$$|t|^j |(D^k b^{-1})(t)| \leq |t|^j \frac{C_k'(m(p)+2)^{p|k|}}{C^{|k|+1}\ (m(p)-2)^{(p-1)(|k|+1)}} \leq$$

$$\leq C_k''(1 + \frac{4}{m(p)-2})^{p|k|+j} \cdot (m(p)-2)^{-p+j+|k|+1} \leq$$

$$\leq C_k''(1+ \frac{4}{p})^{p|k|+j} \cdot (m(p)-2)^{-p+j+|k|+1}.$$

The last expression is bounded and so the proof is complete.

2.1. <u>Theorem</u>. Let K be a bounded subset of $\mathcal{S}$. Given $\varepsilon > 0$, s_o natural and $j_o \geq 0$, $k_o = (k_{o1}, \dots, k_{on}) \geq 0$, there exists an a in $\mathcal{S}$ and a sequence $(K_s)_{s=1}^{\infty}$ of bounded subsets of $\mathcal{S}$ with the following property: for $x \in K$ there exists an $y_s \in K_s \cap (\mathcal{S} x)^-$ with

1° $x = a^s y_s$ for $s = 1,2,\dots$

2° $|x-y_s|_{j_o k_o} \leq \varepsilon$ for $s = 1,2,\dots,s_o$.

Moreover, if $x_n \longrightarrow 0$ in $\mathcal{S}$ then the corresponding $y_{sn} \xrightarrow[n\to\infty]{} 0$ for each s.

Proof. The subset K is bounded, so there are a positive function h and nonnegative constants $(q_k)_{k\geq 0}$ such that $\sup_{t\in R_n} |t|^j h(t) < M_j < \infty$ for all $j \geq 0$ and $|(D^k x)(t)| \leq q_k h(t)$ for all $t \in R_n$, $k \geq 0$, $x \in K$ (see [10], p. 235). Denote by $Q_p = \max_{|k|\leq p} q_k$ and take a sequence $(\varepsilon_p)_{p=1}^{\infty}$ of positive numbers. Since $D^k w^r$ is a polynomial in indeterminates $|t|$, $|t|^{-1}$, $t_1, \dots, t_n$ $(t \neq 0)$ it follows from (2) that we can find a sequence $(m(p))_{p=1}^{\infty}$ such that

(i) $|t|^j|(D^k w^r)(t)|\ h(t) \le \varepsilon_p$ for $|t| \ge m(p)-2$ and all $|k| \le p$, $0 \le r \le p^2$, $j \le p(p+1)$

(ii) $|t|^{j_o+s_o p}|(D^k w^r)(t)|\, h(t) \le \varepsilon_p$ for all $p = 1,2,\dots,$ $0 \le r \le s_o p$, $0 \le k \le k_o$, $|t| \ge m(p)-2$

(iii) $m(p+1)-m(p) \ge 5$, $m(1) \ge 3$

It follows that

(6) $|t|^j|(D^k w^r)(t)|\ |(D^i x)(t)| \le Q_p\, \varepsilon_p$

for $|t| \ge m(p)-2$, $|i|, |k| \le p$, $0 \le r \le p^2$, $j \le p(p+1)$, $x \in K$ and

(7) $|t|^{j_o+s_o p}|(D^k w^r)(t)|\ |(D^i x)(t)| \le Q_{k_o}\, \varepsilon p$

for $|t| \ge m(p)-2$, $0 \le i,k \le k_o$, $0 \le r \le s_o p$, $x \in K$ and $p = 1,2,\dots$

Let us take an $x \in K$ and consider the factorization of x in the form $x = b^{-s}(b^s x)$ where be is the function corresponding to $(m(p))_{p=1}^{\infty}$ according to Lemma 1.1. The function b^{-1} belongs to $\mathcal{S}$ so that b^{-s} belongs to $\mathcal{S}$ as well. We have to show that $b^s x$ is in $\mathcal{S}$ ($s = 1,2,\dots$), i.e.

$$\sup_p \max_{m(p)-2 \le |t| \le m(p+1)-2} |t|^j|(D^k b^s x)(t)| < \infty \text{ for all } j \ge 0,$$

$s \ge 1$, $k \ge 0$. Having known that $b^s x \in \mathcal{S}$ it follows that $b^s x \in (\mathcal{S} x)^-$. Indeed, there exists an approximate unit $(e_n)_{n=1}^{\infty}$ in $\mathcal{S}$ consisting of functions with compact supports, so that $b^s x = \lim e_n b^s x$ for $x \in \mathcal{S}$, $s = 1,2,\dots$. Since $e_n b^s$ are in $\mathcal{S}$ as functions with compact supports, we obtain $b^s x \in (\mathcal{S} x)^-$.

Fix j,k and s. If $p > \max(|k|,j,s)$ we obtain according to (6), for $|t| \in \langle m(p)+2, m(p+1)-2 \rangle$, the estimate

$|t|^j|(D^k(b^s x))(t)| = |t|^k|(D^k(w^{sp}x))(t)| =$

$$=|t|^j \sum_{0\le i\le k} \binom{k}{i} |(D^i w^{sp})(t)|\ |(D^{k-i}x)(t)| \le 2^{|k|} Q_p \varepsilon_p.$$

Now, assume $|t| \in \langle m(p)-2, m(p)+2\rangle$. It is easy to see that, for $f \in C^\infty(R_n)$, $D^k f^s$ is a polynomial $P_{k,s}$ of order s in indeterminates $f,\dots,D^k f$. Again, according to (6), $|t|^{p^2+p}$. $|(D^{k-i}x)(t)| \le Q_p\varepsilon_p$. This, together with 2° of 1.1 yields

$$|t|^j |(D^k b^s x)(t)| \le |t|^j \sum_{0\le i\le k} \binom{k}{i} |(D^i b^s)(t)|\ |(D^{k-i}x)(t)| =$$

$$= \sum_{0\le i\le k} \binom{k}{i} |P_{i,s}(b(t),\dots,(D^i b)(t))|\ |t|^j |(D^{k-i}x)(t)| \le$$

$$\le \sum_{0\le i\le k} \binom{k}{i} K_{i,s}\ (m(p)+2)^{sp}\ |t|^{j-p^2-p} |t|^{p^2+p} |(D^{k-i}x)(t)| \le$$

$$\le (m(p)+2)^{sp}\ (m(p)-\ell)^{-p^2}\ .\ Q_p\varepsilon_p \sum_{0\le i\le k} \binom{k}{i} K_{i,s} \le$$

$$\le \left(1+\frac{4}{m(p)-2}\right)^{sp} \sum_{0\le i\le k} \binom{k}{i} K_{i,s}\, Q_p\varepsilon_p,$$ where $K_{i,s}$ are suitable constants. From the last two estimates we obtain

$|t|^j |(D^k b^s x)(t)| \le M_{k,s}\ .\ Q_p\varepsilon_p$ for $|t| \in \langle m(p)-2, m(p+1)-2\rangle$, p large enough. According to (7) we obtain, for $s=1,2,\dots,s_o$, $p=1,2,\dots$, similar estimates

$|t|^{j_o} |(D^{k_o} b^s x)(t)| \le M\varepsilon_p$, where M, $M_{k,s}$ are suitable constants.

Given an $\varepsilon > 0$, let us choose now $(\varepsilon_p)_{p=1}^\infty$ so that $Q_p\varepsilon_p \longrightarrow 0$ and $\max_{p\in N} M\varepsilon_p \le \varepsilon/2$. Using these estimates we can deduce the following facts. First, $b^s x \in \mathcal{S}$ for $x \in K, s=1,2,\dots$, all $b^s K = K_s$ are bounded in $\mathcal{S}$ and $|x-b^s x|_{j_o k_o} = \max_{t\in R_n} |t|^{j_o}$ $|(D^{k_o}x)(t)-(D^{k_o}b^s x)(t)| = \max_{|t|\ge m(1)-2} |t|^{j_o} |(D^{k_o}x)(t)-(D^{k_o}b^s x)(t)| \le$ $\le \varepsilon$ for $s=1,2,\dots,s_o$. Finally, if $x_n \longrightarrow 0$ in $\mathcal{S}$ then, for $K = (x_n)_{n=1}^\infty$, we obtain $y_n = b^s x_n$ tends to zero as well. Indeed, let us fix j, k and s. Given an $\varepsilon > 0$, let us find

p_o so that $M_{k,s}Q_p \varepsilon_p \le \varepsilon$ for $p \ge p_o$. There exists n_o such that, for $n \ge n_o$, $|x_n|_{jl} \le (\sum_{0 \le i \le l} \binom{l}{i} \sup_{|t| \le m(p_o)} |(D^i b^s)(t)|)^{-1}$ for $l \le k$. It follows that, for $n \ge n_o$, we have

$$|y_n|_{jk} = \max (\max_{|t| \le m(p_o)} |t|^j |(D^k b^s x_n)(t)| +$$

$$\max_{|t| \ge m(p_o)} |t|^j |(D^k b^s x_n)(t)|) \le \varepsilon$$

The proof is complete.

Remark. Since the Fourier transformation is a continuous linear mapping of $\mathcal{S}$ onto itself and takes the pointwise multiplication to the convolution, Theorem 2.1 holds also if we replace the multiplication by the convolution.

Acknowledgment. The authors wish to express their gratitude to J. Fuka, V. Müller and M. Zajac for their kind help during the final preparation of the paper.

References

[1] G.R. ALLAN, A.M. SINCLAIR: Power factorization in Banach algebras with a bounded approximate unit, Studia Math. 56(1976), 31-38

[2] E. CHRISTIANSEN: On factorization in Fréchet algebras (unpublished)

[3] P.J. COHEN: Factorization in group algebras, Duke Math. J. 26(1959), 199-205

[4] I.G. CRAW: Factorisation in Fréchet algebras, J. London Math. Soc. 44(1969), 607-611

[5] P.C. CURTIS, A. FIGA-TALAMANCA: Factorization theorems for Banach algebras, in Function Algebras, ed. F.T. Birtal, Scott, Foresman and Co. Chicago 1966

[6] J. KŘÍŽKOVÁ, P. VRBOVÁ: A remark on a factorization theorem, Comment. Math. Univ. Carolinae 15 (1974), 611-614

[7] V. PTÁK: Deux théorèmes de factorisation, Comptes Rendus Ac. Sci. Paris 278(1974), 1091-1094

[8] V. PTÁK: A theorem of the closed graph type, Manuscripta Math. 13(1974), 109-130

[9] V. PTÁK: Factorization in Banach algebras, Studia Math. 65,2 (in print)

[10] L. SCHWARTZ: Théorie des distributions, Hermann, Paris, 1966

Matematický ústav ČSAV

Žitná 25

11567 Praha 1

Československo

(Oblatum 12.5. 1978)