Donald E. Bennett
Expansive collections of continua

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Abstract: Let $X$ be a continuum. A collection $F$, of proper subcontinua of $X$ is said to be expansive provided that if $F \subseteq T$ and $G$ is a proper subcontinuum of $X$ such that $F \subseteq G$, then $G \in F$. In this paper such collections of subcontinua are studied. In particular, if $X$ is the union of the members of $F$ then conditions are given which imply that $X$ can be written as the union of two members of $F$.

Key words and phrases: Continuum, expansive collections, indecomposable, irreducible, non-separating subcontinua.

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In this paper certain collections of proper subcontinua of a continuum $X$ are studied. In particular, those collections which "expand" with respect to set inclusion are investigated and properties of such collections are developed. If $X$ is the union of the subcontinua from such a collection, then conditions are given which imply that $X$ is the union of exactly two subcontinua from the collection.

Throughout this paper the continuum $X$ is a compact connected metric space. The continuum $X$ is said to be decomposable if it is the union of two proper subcontinua; otherwise, the continuum is indecomposable. If $K$ is a proper subcontinuum of $X$, then $K$ is non-separating in $X$ means that
X - K is connected. The continuum X is irreducible if there are two points p and q in X such that no proper subcontinuum of X contains both p and q. If A is a subset of X, then the closure of A in X will be denoted by Ȳ. For terms and notation used but not defined herein, the reader is referred to [3].

**Definition:** A collection, $\mathcal{F}$, of proper subcontinua of X is said to be expansive provided if $F \in \mathcal{F}$ and G is a proper subcontinuum of X such that $F \subseteq G$, then $G \in \mathcal{F}$.

Let $S \subseteq X$, $S \neq \emptyset$, and $\mathcal{F}$ be the collection of all proper subcontinua of X that contain S. Then $\mathcal{F}$ is an expansive collection.

A proper subcontinuum K of X is said to be a terminal continuum provided if A and B are proper subcontinua of X such that $X = A \cup B$ and $A \cap K \neq \emptyset$ $B \cap K$ then $X = A \cup K$ or $X = B \cup K$ [1]. The terminal subcontinua of X form an expansive collection of non-separating subcontinua of X.

If $\mathcal{F}$ is an expansive collection of subcontinua of X, then we shall let $\mathcal{F}^* = \bigcup \{ F \mid F \in \mathcal{F} \}$.

It is easily seen that $\mathcal{F}^*$ is dense in X and is non-separating in X. Moreover, if $X - \mathcal{F}^*$ is a non-empty subcontinuum then $X - \mathcal{F}^*$ does not separate any subcontinuum of X. But when is $X - \mathcal{F}^*$ a continuum? The following theorem provides an answer.

**Theorem 1.** Suppose $\mathcal{F}$ is an expansive collection of proper subcontinua of X. Then $X - \mathcal{F}^*$ is a continuum if and only if the only subcontinua that intersect both $\mathcal{F}^*$ and $X - \mathcal{F}^*$ are decomposable.
Proof: Assume that $X - F^*$ is not a continuum. Then $X - F^* \neq X - F^*$ . Thus $X - F^*$ is a continuum which intersects both $F^*$ and $X - F^*$ , hence is decomposable. Let $A$ and $B$ be proper subcontinua such that $X - F^* = A \cup B$. It follows that $X - F^* \neq A$ and $X - F^* \neq B$. Now either $A$ or $B$ intersects $F^*$ so without loss of generality assume that $A \cap F^* \neq \emptyset$. Let $F \in \mathcal{F}$ such that $A \cap F \neq \emptyset$. Then $A \cup F$ is a continuum.

If $A \cup F = X$, then $X - F^* \subset X - F$. Since $X - F \subset A$ this would imply that $X - F^* \subset A$ which is not the case. Thus $A \cup F$ must be a proper subcontinuum of $X$. Since $F \subset A \cup F$ then $A \cup F \in \mathcal{F}$ and it follows that $A \subset F^*$. This implies that $X - F^* \subset B$ which is a contradiction. Therefore $X - F^*$ is, in fact, a continuum.

Now suppose that $X - F^*$ is a continuum but that $K$ is an indecomposable subcontinuum which intersects $F^*$ and $X - F^*$. Let $F_\infty \in \mathcal{F}$ such that $K \cap F_\infty \neq \emptyset$, then $K \cup F_\infty$ is a subcontinuum of $X$ which contains $F_\infty$. Since $K \neq F^*$, it follows that $K \cup F_\infty \notin \mathcal{F}$ which implies that $X = K \cup F_\infty$. Note that $X - F^* \subset X - F_\infty \subset K$. Let $C$ be the component of $K$ which contains $X - F^*$. Since $X - F^* \subset C$, there is a $F_\beta \in \mathcal{F}$ such that $C \cap F_\beta \neq \emptyset$. Let $I$ be a subcontinuum contained in $C$ such that $I \cap (X - F^*) \neq \emptyset$. Now $I \cup F_\beta$ is a subcontinuum which contains $F_\beta$ but $I \cup F_\beta \notin \mathcal{F}$. Since $\mathcal{F}$ is an expansive collection of proper subcontinua of $X$, it follows that $X = I \cup F_\beta$. Thus $K - I \subset F_\beta$ which implies that $K \cap F_\beta \neq \emptyset$. Since $X - F^* \subset K$, then $X - F^* \subset F_\beta$ which is a contradiction. Therefore the only subcontinua that intersect both $F^*$ and $X - F^*$ are decomposable.
Obviously, \( X - \mathcal{F}^* \) is a continuum if the continuum \( X \) is hereditarily decomposable. It also holds with a somewhat weaker condition on \( X \).

**Corollary 1:** Suppose \( X \) is a continuum such that each indecomposable subcontinuum has void interior and \( \mathcal{F} \) is an expansive collection of proper subcontinua of \( X \). Then \( X - \mathcal{F}^* \) is a continuum.

**Proof:** Let \( K \) be a continuum which intersects both \( \mathcal{F}^* \) and \( X - \mathcal{F}^* \). Let \( F \in \mathcal{F} \) such that \( K \cap F \neq \emptyset \). Then \( K \cup F \) is a continuum which contains \( F \) but \( K \cup F \notin \mathcal{F} \). Thus \( X = K \cup F \) and \( X - F \) is an open set contained in \( K \). By hypothesis \( K \) is decomposable and the corollary follows from the previous theorem.

Continua which satisfy the hypothesis of the following theorem are equivalent to the "type A" continua of Thomas [2].

**Theorem 2.** Let \( X \) be an irreducible continuum such that each indecomposable subcontinuum has void interior. If \( X = \mathcal{F}^* \), then there exists \( F_\alpha \in \mathcal{F} \) and \( F_\beta \in \mathcal{F} \) such that \( X = F_\alpha \cup F_\beta \).

**Proof:** Suppose \( \{ p, q \} \subseteq X \) such that \( X \) is irreducible between \( p \) and \( q \). Let \( \mathcal{F}_p = \{ F \in \mathcal{F} \mid p \in F \} \) and \( \mathcal{F}_q = \{ F \in \mathcal{F} \mid q \in F \} \). Then \( \mathcal{F}_p \) and \( \mathcal{F}_q \) are non-empty expansive collections of subcontinua and, according to Corollary 1, \( X - \mathcal{F}_p^* \) and \( X - \mathcal{F}_q^* \) are continua. Note that \( p \notin X - \mathcal{F}_q^* \) and \( q \notin X - \mathcal{F}_p^* \).

Case 1. If \( \mathcal{F}_p^* \cap \mathcal{F}_q^* \neq \emptyset \), let \( F_\alpha \in \mathcal{F}_p \) and \( F_\beta \in \mathcal{F}_q \) such that \( F_\alpha \cap F_\beta \neq \emptyset \). Since \( F_\alpha \cup F_\beta \) is a subcon-

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tinum containing \( \{p, q\} \), it follows that \( X = E_\alpha \cup F_\beta \).

**Case 2.** If \( \mathcal{F}_p^* \cap \mathcal{F}_q^* = \emptyset \). Then \( X = (X - \mathcal{F}_p^*) \cup (X - \mathcal{F}_q^*) \). Let \( F \in \mathcal{F} \) such that \( p \in F \). Then \( F \cap \mathcal{F}_q^* = \emptyset \) which implies that \( F \subset X - \mathcal{F}_q^* \). Since \( X - \mathcal{F}_q^* \) is a proper subcontinuum of \( X \) it follows that \( X - \mathcal{F}_q^* \in \mathcal{F} \).

Likewise \( X - \mathcal{F}_p^* \in \mathcal{F} \). So \( X = E_\infty \cup F_\beta \) where \( E_\infty = X - \mathcal{F}_q^* \) and \( F_\beta = X - \mathcal{F}_p^* \).

In the remaining portion of this paper we shall assume that the expansive collection \( \mathcal{F} \) of subcontinua of \( X \) has the property that if \( E_\infty \in \mathcal{F} \) and \( F_\beta \in \mathcal{F} \) then \( E_\infty \cap F_\beta = \emptyset \).

**Lemma 1:** There is a countable subcollection \( \mathcal{F}_\sigma^* \) of \( \mathcal{F} \) such that \( \mathcal{F}_p^* = \mathcal{F}_\sigma^* \).

**Proof:** Case 1: If there exist \( F_1 \in \mathcal{F} \) and \( F_2 \in \mathcal{F} \) such that \( X = F_1 \cup F_2 \), let \( \mathcal{F}_\sigma^* = \{F_1, F_2\} \). Then \( \mathcal{F}_p^* = F_1 \cup F_2 = \mathcal{F}_\sigma^* \).

Case 2. Suppose that \( X \) is not the union of two members of \( \mathcal{F} \). Choose a \( E_\infty \in \mathcal{F} \) and let \( \{V_i\}_{i=1}^{\infty} \) be a countable basis for \( X - E_\infty \). For each positive integer \( i \), let \( L_i \) be the component of \( X - V_i \) which contains \( E_\infty \). Since \( \mathcal{F} \) is an expansive collection then each \( L_i \in \mathcal{F} \). Let \( \mathcal{F}_\sigma^* = \{L_i\}_{i=1}^{\infty} \). Then \( \mathcal{F}_\sigma^* \subset \mathcal{F} \) which implies that \( \mathcal{F}_p^* \subset \mathcal{F}_\sigma^* \).

Suppose \( x \in \mathcal{F}_p^* \). Let \( F_\beta \in \mathcal{F} \) such that \( x \in F_\beta \). Now by hypothesis \( E_\infty \cup F_\beta \) is a subcontinuum of \( X \). Since \( E_\infty \cup F_\beta \neq X \), then \( W = X - (E_\infty \cup F_\beta) \) is an open subset of \( X - E_\infty \).

There is a positive integer \( i \) such that \( V_i \subset W \) which implies that \( E_\infty \cup F_\beta \subset X - V_i \). Thus \( E_\infty \cup F_\beta \subset L_i \) so \( x \in F_\beta \subset L_i \subset \mathcal{F}_p^* \). Therefore \( \mathcal{F}_p^* \subset \mathcal{F}_\sigma^* \) and it follows that \( \mathcal{F}_p^* = \mathcal{F}_\sigma^* \).

**Theorem 3:** Let \( \mathcal{F} \) be an expansive collection of non-separating subcontinua of \( X \). If \( \mathcal{F}_p^* = X \), then there are
Proof: Suppose to the contrary that $F^* = X$ but $X$ is not the union of any two numbers of $F$ . Then it follows that $X$ is not the union of any finite subcollection of $F$ .

Let $F = \{ F_i \}_{i=1}^{\infty}$ be a countable subcollection of $F$ such that $F^* = F^*$. For each positive integer $n$ let $N_n = \bigcup_{i=1}^{\infty} F_i$. Then $\{ N_i \}_{i=1}^{\infty}$ is an increasing sequence of proper subcontinua of $X$ with $\bigcup_{i=1}^{\infty} N_i = X$.

Assertion: For each integer $i$, there is a $j > i$ such that $X - N_j \subseteq X - N_i$. For if not, then there is an $i$ such that for all $j > i$ $X - N_j = X - N_i$. Thus $\{ X - N_j \}_{j>i}$ is a countable collection of open sets, each dense in $X - N_i$.

According to the Baire Category Theorem, there is a point $p \in \bigcup_{j>i} (X - N_j) = X - \bigcup_{j>i} N_j$. But this would imply that $p \in X - \bigcup_{i=1}^{\infty} N_i$ which is a contradiction. Therefore the assertion holds.

Thus we may obtain a subsequence $\{ N'_i \}_{i=1}^{\infty}$ of the sequence $N$ such that for each $i$, $N'_i \subset N'_{i+1}$ while $X - N'_i \supseteq X - N'_i$. Clearly $X = \bigcup_{i=1}^{\infty} N'_i$. Since $\{ X - N'_i \}_{i=1}^{\infty}$ is a decreasing sequence of compact sets, there is an $x \in \bigcap_{i=1}^{\infty} (X - N'_i)$. Let $j$ be a positive integer such that $x \in N'_j$. Then $x \in X - N'_j$ and $[N'_j \cup X - N'_{j+1}]$ is a proper subcontinuum of $X$. Since $[N'_j \cup X - N'_{j+1}]$ does not separate $X$, it must be a member of $F$ . Thus $N'_{j+1}$ and $[N'_j \cup X - N'_{j+1}]$ are two members of $F$ whose union is $X$. This contradiction establishes the theorem.

Theorem 3 was suggested by J.B. Fugate.
References


Department of Mathematics
Murray State University
Murray, KY, 42071
U.S.A.

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