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MEASURE THEORETIC BEHAVIOR OF CLOSED SETS

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Abstract: In this paper we introduce the idea of a P_1-set. A closed set P is a P_1-set if for any positive regular Borel measure m, $F \cap$ support $(m) \neq \emptyset$ implies $m(F) > 0$. Every P-set is a P_1-set, but it is unknown whether every P_1-set is a P-set.

Key words: P-set, P_1-set, P-point, Borel measure, extremally disconnected space.

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1. P_1-sets. Throughout, X will be a compact T^2 space. If $m \in M^+$, where M^+ is the set of positive regular Borel measures on X, then $S = S(m)$ is the closed support set of m. We recall that a closed set F is a P-set if its neighborhood system is closed under countable intersections. P-sets have the following interesting property: if $m \in M^+$, then $S \cap \cap F \neq \emptyset$ implies $m(F) > 0$. Let us call a closed set having this property a P_1-set. In this section we give a number of equivalent characterizations of P_1-sets, and a result on compact spaces with the property that the closure of a cozero set is always a P_1-set. This generalizes the corresponding result of Seever [S] for F-spaces (where the closure of a cozero-set is always a P-set).

We do not know if there exist P_1-sets which are not

P-sets. We show in section 2 that if they exist anywhere, they can "usually" be embedded in $\beta N \setminus N$.

Theorem 1. For a closed set F, the following are equivalent: (1) F is a P_1-set, (2) for all $m \in M^+$, $S \cap F$ is clopen in S, (3) for all $m \in M^+$, $S \cap F$ is either empty or a P''-set, i.e., $S \cap F \subset Z$ (zero set) implies $\mathrm{int}_S (Z \cap S) \neq \emptyset$, (4) for all $m \in M^+$, support (m_F) = support $(m) \cap F$, (5) for all $m \in M^+$ and all open V, $V \cap F \cap S \neq \emptyset$ implies $m(V \cap F) > 0$, (6) if $S = \bigcup S_n$, where the S_n are support sets, then $(\mathrm{cl}_X S) \cap F = \mathrm{cl}_F (S \cap F)$.

Proof. The pattern will be $1 \to 2 \to 3 \to 1$, and $6 \to 1 \to \to 5 \to 4 \to 5 \to 6$.

(1) implies (2). Let $V = S \setminus (F \cap S)$, an open set in S. Let $n = m_V$, i.e., n is the regular Borel measure defined by $n(A) = m(A \cap V)$. Then $n(F) = 0$. Now support $(n) = \mathrm{cl}_S V$. [For if $x \in \mathrm{cl}_S V$ and W is an S-open neighborhood of x, then $W \cap V \neq \emptyset$, and hence $n(W) = m(W \cap V) > 0$.] By (1), $F \cap \mathrm{cl}_S V = \emptyset$. Thus $F \cap S$ is open, as well as closed, in S.

(2) implies (3). Obvious.

(3) implies (1). Suppose $S \cap F \neq \emptyset$, while $m(F) = 0$. By regularity there exists a descending sequence of S-open sets V_n with $F \cap S \subset V_n$ and $m(V_n) \to 0$. Since F is closed we may assume $\mathrm{cl}\, V_{n+1} \subset V_n$. Then $m(\bigcap V_n) = 0$, whence $\mathrm{int}_S (\bigcap V_n) = \emptyset$, contrary to the assumption that $F \cap S$ is a P''-set.

(6) implies (1). Suppose (1) fails, so that for some $S = S(m)$ we have $F \cap S \neq \emptyset$, but $m(F) = 0$. Let A_n be an ascending sequence of compact subsets of $S \setminus F$ such that $m(A_n) \to \to m(S)$. Define measures m_n by the formula $m_n(B) = m(B \cap A_n)$. Then S_n = support $(m_n) \subset A_n$. Let $T = \bigcup S_n$. Then

$cl_F(T\cap F) = cl_F(\emptyset) = \emptyset$. However, $cl_X T = cl_X \cup S_n = S$, where S = support (m). [For if not, then there is an open set V such that $m(V) = t > 0$, whence $m(A_n) < m(S) - t$, contrary to $m(A_n) \longrightarrow m(S)$.] Thus, $(cl_X T)\cap F = S\cap F \neq \emptyset$, while $cl_F(T\cap F) = \emptyset$. Hence (6) fails.

(1) implies (5). If (5) fails, then there is an open V such that $m(V\cap F) = 0$, while $V\cap F\cap S \neq \emptyset$. Let $n = m_V$. Then support $(n) \supset V\cap S$; so (1) fails, since $F\cap$ support $(n) \neq \emptyset$, while $n(F) = 0$.

(5) implies (4). Clearly, the left side of the formula in (4) is contained in the right side. For the reverse inclusion, let $x \in$ support $(m)\cap F$. If V is a neighborhood of x, then $m_F(V) = m(V\cap F) > 0$, and hence $x \in$ support (m_F).

(4) implies (5). If $V\cap F\cap S \neq \emptyset$, let x be in this set. Then $x \in$ support (m_F), so $0 < m_F(V) = m(V\cap F)$.

(5) implies (6). Clearly, $cl_F(S\cap F) \subset (cl_X S)\cap F$. For the reverse inclusion, suppose $x \notin cl_F(S\cap F)$. Then there is an open V containing x such that $V\cap (S\cap F) = \emptyset$. Then $V\cap (S_n\cap F) = \emptyset$ for all n, whence $m(V\cap F) = 0$, where

$$m = \sum 2^{-n} \| m_n \|^{-1} m_n .$$

By (5), $\emptyset = V\cap F\cap$ support $(m) = V\cap F\cap cl_X S$. Since $x \in V$, it follows that $x \notin F\cap cl_X S$.

Remark. If in condition (6) we allow the S_n to be arbitrary compact sets, we get a characterization of P-sets. We leave details to the reader.

Theorem 2. Let X be a compact T^2 space such that the closure of a cozero set is always a P_1-set. Then any support set $S = S(m)$ is extremally disconnected in its subspace topology.

Proof. It suffices to prove S is an F-space, since an F-space with countable chain condition is extremally disconnected. Let A_0 and B_0 be disjoint cozero sets in S. As in [Sem, page 432], we may write $A_0 = \{x : f(x) > 0\}$ and $B_0 = \{x : f(x) < 0\}$ for some $f \in C(S)$. Let g be any element of $C(X)$ which extends f. If $A = \{x : g(x) > 0\}$ and $B = \{x : g(x) < 0\}$, then $A_0 = A \cap S$ and $B_0 = B \cap S$. Define a measure $n = m_A$. Then support $(n) = cl_S A_0$. Let $F = cl_X B$. Since $F \cap A_0 = \emptyset$, we have $n(F) \neq 0$. Since F is a P_1-set, it follows that $\emptyset = F \cap \text{Support}(n) = cl_X B \cap cl_S A_0 \supset cl_S B_0 \cap cl_S A_0$. Thus, disjoint cozero sets in S have disjoint closures, i.e., S is an F-space.

Corollary. If X is as in the theorem, then C(X) is a Grothendieck space.

Proof. The proof of theorem 2.2 of [S] shows that if every support set is extremally disconnected, then C(X) is a G-space.

Remark. In [I-S] it is asked what sorts of Borel sets have the "Grothendieck property", i.e., how to characterize sets such that if m_n and m are Borel measures with $m_n \longrightarrow m$ weak-*, then $m_n(F) \longrightarrow m(F)$. P_1-sets satisfy the somewhat stronger conclusion that $m_n|_F \longrightarrow m|_F$ weak-*. In fact, in the last assertion, sequences may be replaced by countable nets. (Perhaps this is a property which characterizes P_1-sets among the closed sets.)

2. The existence problem. We do not know whether the-

re exist P_1-sets which are not P-sets, but theorem 3 below may be helpful in this respect. The special case of P_1-points merits special interest. A P_1-point is one which does not belong to the support of any element of M^+ such that $m(p) = 0$. In [K] it is shown that in $\beta N \setminus N$ there exist points which are not P-points, and which are not points of accummulation of any countable subset. Let us call these P_2-points. It is easy to see that P-points $\subset P_1$-points $\subset$ $\subset P_2$-points. Assuming CH, at least one of these inclusions is proper in the $\beta N \setminus N$ case, but it is not known which.

<u>Lemma 1</u>. Let $f: X \longrightarrow Y$ be continuous, where X and Y are compact T^2. If K is a P_1-set in Y, then $f^{-1}(K)$ is a P_1-set in X.

Proof. Let m be a positive regular Borel measure on X, and suppose $m(f^{-1}K) = 0$. Define a positive regular Borel measure m_o on Y by $m_o(A) = m(f^{-1}A)$. Then $m_o(K) = 0$, so $K \cap$ $\cap$ support $(m_o) = 0$, since K is a P_1-set. If $V = Y \setminus$ $\setminus$ support (m_o), then V is an open set with $K \subset V$ and $m_o(V) =$ $= 0$. Now $f^{-1}K \subset f^{-1}V$, where $f^{-1}V$ is open, and $m(f^{-1}V) =$ $= m_o(V) = 0$. Thus, $f^{-1}K \cap$ support $(m) = \emptyset$, so $f^{-1}K$ is a P_1-set.

<u>Lemma 2</u> [V, theorem 8]. Let $f: X \longrightarrow Y$ be continuous and onto, and K be a closed subset of Y. If $f^{-1}K$ is a P-set, then K is a P-set. (The converse also holds, will not be needed here.)

The rather easy proof, which is omitted in [V], is left as an exercise.

<u>Theorem 3</u>. [CH] Let X be a compact T^2 space such that

the cardinality of the open sets is c. If X contains a P_1-set which is not a P-set, then $\beta N \setminus N$ contains a P_1-set which is not a P-set.

Proof. Let K be such a set in X. If E(X) is the Gleason space of X and f the Gleason map, then $L = f^{-1}K$ is a P_1-set in E(X), by lemma 1. It is shown in [K] that under CH an extremally disconnected space with c open sets can be embedded as a P-set in $\beta N \setminus N$. By lemma 2, L is not a P-set in E(X), and it is easy to check that it is not a P-set in $\beta N \setminus N$ either. To show that L is a P_1-set in $\beta N \setminus N$, let m be a positive regular Borel measure on $\beta N \setminus N$ with support set S, and suppose $S \cap L \neq \emptyset$. Let $n = m_{E(X)}$. Since E(X) is a P-set (hence also a P_1-set) in $\beta N \setminus N$, $n > 0$ and condition (4) of theorem 1 implies that support (n) = $S \cap E(X)$. n defines a regular positive Borel measure on E(X). Since support $(n) \cap L \neq \emptyset$ and L is a P_1-set in E(X), we have $m(L) = n(L) > 0$. Hence L is a P_1-set in $\beta N \setminus N$.

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