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ON HOMOTOPICAL EQUIVALENCE OF TOLERANCE SPACES

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Abstract: It is shown that in the category of tolerance spaces the homotopical equivalence can be described by inner structure of these spaces.

Key words: Homotopical equivalence, tolerance spaces.

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§ 1. Introduction. The present paper considers the class of finite symmetric graphs with loops - tolerance spaces - and their homotopy. The homotopy equivalence of tolerance spaces is defined similarly to that of topological spaces. The interval I is for our purposes substituted by suitable tolerance space I_n (see the definition 3.1) and continuous mappings by mappings preserving relations.

The principal result of this paper is a characterization of the homotopical equivalence between two tolerance spaces by their inner structure, i.e. without using homomorphisms.

Though the paper was stimulated by ideas comprised in [1], the adopted terminology is slightly different from that used in [1]. Especially instead of an N-space and an N-map we are going to use terms a tolerance space and a homomorphism, which are more common nowadays. (See [2].)

§ 2. Tolerance spaces

2.1. Definition. A tolerance space (X,R) is a non-void finite set X endowed with a reflexive symmetrical relation R. A homomorphism $f:(X,R) \longrightarrow (Y,S)$ is a mapping from X to Y such that $(x,y) \in R$ implies $(f(x),f(y)) \in S$. The identity homomorphism of a space X will be often denoted by 1_X.

2.2. Conventions. i) If there is no danger of confusion we are going to speak about a space instead of a tolerance space and write simply X instead of (X,R), etc..

ii) The elements of tolerance spaces will be frequently called points.

iii) We will write $st_{(X,R)}x$ for the set $\{y;(x,y) \in R\}$. The subscript will be often omitted.

iv) The fact that two spaces X, Y are isomorphic (i.e., that there are homomorphisms f, g such that $fg = 1_Y$ and $gf = 1_X$) will be indicated by $X \cong Y$.

2.3. The category of tolerance spaces and their homomorphisms will be denoted by Tol.
The product $(X,R) \times (Y,S)$ in Tol is obviously obtained as $(X \times Y,T)$ with $T = \{((x,u),(y,v));\ (x,y) \in R \text{ and } (u,v) \in S\}$.

2.4. Further conventions. (X,R) together with a 1-1 homomorphism $i:(X,R) \longrightarrow (Y,S)$ is called a subspace of (Y,S) if $(i(X);(i \times i)(R)) = (i(X);\ (i(X) \times i(X)) \cap S)$.
The union of (X,R) and (Y,S) is defined as $(X \cup Y,\ R \cup S)$.

§ 3. Homotopy, contractible points and a center of a space

3.1. Definition. Let us denote by I_n the set $\{1,\dots,n\}$

with the tolerance relation $R=\{(i,j); |i - j| \leq 1\}$. Homomorphisms $f,g:X \to Y$ are said to be homotopical if there is an $n \in N$ and a homomorphism $F_n: X \times I_n \to Y$ such that $F_n(-,1) = f(-)$ and $F_n(-,n) = g(-)$. The homotopy between f, g will be denoted $f \sim g$ or $F_n: f \sim g$ if we want to stress the mapping F_n.

We will declare tolerance spaces X, Y as homotopically equivalent if there exist homomorphisms $f:X \to Y$, $g:Y \to X$ such that $fg \sim 1_Y$, $gf \sim 1_X$. A homotopical equivalence between X, Y will be indicated by $X \sim Y$.

3.2. Remark. We immediately see that the relation "to be homotopically equivalent" on tolerance spaces is indeed an equivalence.

3.3. <u>Definition</u>. A point $x \in X$ for which there exists an element $y \in X$, $y \neq x$ such that $st\, x \subset st\, y$ is said to be contractible. Let $x \in (X,R)$ be a contractible point. Then the space (X',R') with $X' = X - \{x\}$ and $R' = R \cap (X' \times X')$ is called a contracted subspace of (X,R). For every space X an $n \geq 0$ and a sequence of spaces $\{X_i\}_{i=0}^{n}$ can be assigned such that $X_0 = X$, X_i is a contracted subspace of X_{i-1} and X_n contains no contractible points. Let us call X_n centre of X and denote it by $K(X)$. The $\{X_i\}_{i=0}^{n}$ will be called a centralising sequence. Each centralising sequence $\{X_i\}_{i=0}^{n}$ determines a sequence $\{x_i\}$ of contractible points where x_i is contractible in X_i and $X_{i+1} = X_i \setminus \{x_i\}$.

3.4. <u>Lemma</u>. If u, v are two contractible points of a space (Y,S) and if we denote Y_u and Y_v respectively the respective contracted spaces then either the u is contractible in Y_v and v in Y_u, or $Y_u = Y_v$. In the second case, if z is contractible in Y_u it is contractible in Y_v as well.

Proof. By the definition of a contractible point we have points $w, z \in (Y,S)$ such that $w \neq u$, $z \neq v$, $st\ u \subset st\ w$ and $st\ v \subset st\ z$. We have the following possibilities: i) $w = z$, ii) $w = v$ and $u \neq z$, iii) $w \neq v$ and $u = z$, iv) $w = v$ and $u = z$.

ad i) It is obvious that the points v and u are contractible points of the spaces Y_u and Y_v respectively.

ad ii) $w = v$ implies $st\ v = st\ w$ and consequently $st\ u \subset st\ z$, $st\ v \subset st\ z$.

ad iii) This is analogous to ii).

ad iv) If there is an $a \in Y$ such that $a \neq u, v$ then $st\ u \subset$ $\subset st\ a$ iff $st\ v \subset st\ a$. Let such an element a exist. Then this case can be transformed to the case i).

Thus, let us assume that such a point a does not exist. Define a homomorphism $f: Y_u \longrightarrow Y_v$ identically on $Y_u \setminus \{v\}$, and put $f(v) = u$. It follows from $w = v$ and $u = z$ that $st\ u =$ $= st\ v$ and f is an isomorphism. On the other hand the isomorphism between Y_u and Y_v yields $st_Y u = st_Y v$.

It remains to be shown that $st_{Y_u} x \subset st_{Y_u} y$ implies $st_Y x \subset st_Y y$. But it is clear because the points u, v are either both elements $st_Y z$ or none of them as follows from the relation $st_Y u = st_Y v$ and thus $st_Y x \subset st_Y y$.

<u>Proposition.</u> Each space X has a centre, and any two of its centres are isomorphic.

Proof: The existence of K(X) is obvious.

Let (X_n, R_n) and (X'_m, R'_m) be two distinct centres of

(X,R). Consider centralising sequences $\{(X_i,R_i)\}_{i=0}^{n}$, $\{(X_i',R_i')\}_{i=0}^{m}$ and the associated sequences $\{x_i\}$, $\{x_i'\}$ of contractible points. Let us define a map from the sequence $\{x_i\}$ to $\{x_i'\}$: A point x_i is being sent into an x_j' such that $x_j' = x_i$ if such a point x_j' exists; if it does not, x_j' is chosen to be a contractible point in the X_i such that leaving it out, a space isomorphic to X_{i+1} is obtained. Using the preceding lemma, we see that the map can be defined for every $i = 0,\dots, n-1$. Moreover it can be chosen 1-1 and onto. Thus $m = n$.

We rearrange the sequences of omitted points for the points included in both sequences to be in the beginning of the new sequences in the same order as in the original ones. The other points of the sequence $\{x_i\}$ put on the remaining places in their original mutual order. Denote the new sequence by $\{y_i\}$. We reset the remaining points of the $\{x_i'\}$ so that for $j = k+1,\dots,n-1$ on the j-th site is the point which is the image of a y_j in the formerly defined mapping. Let us denote the sequence so obtained by $\{y_i'\}$.

The sequences satisfy the following:

$\{y_i;\ i = 0,\dots,n-1\} = \{x_i;\ i = 0,\dots,n-1\}$,

$\{y_i';\ i = 0,\dots,n-1\} = \{x_i';\ i = 0,\dots,n-1\}$,

$\{y_i;\ i = 0,\dots,k\} = \{y_i';\ i = 0,\dots,k\}$.

Moreover, there exist centralising sequences $\{(Y_i,S_i)\}_{i=0}^{n}$ and $\{(Y_i',S_i')\}_{i=0}^{n}$ of (X,R) such that the sequences of contractible points associated with them are precisely $\{y_i\}$ and $\{y_i'\}$. The following holds for these centralising sequences: $(Y_n,S_n) = (X_n,R_n)$, $(Y_n',S_n') = (X_n',R_n')$ and $(Y_k,S_k) = (Y_k',S_k')$.

It is obvious from the lemma and the performed construction that if $(Y_j, S_j) \cong (Y'_j, S'_j)$ for $j = k, \dots, n - 1$ then also $(Y_{j+1}, S_{j+1}) \cong (Y'_{j+1}, S'_{j+1})$. This yields $(Y_n, S_n) \cong (Y'_n, S'_n)$ and thus $(X_n, R_n) \cong (X'_m, R'_m)$.

3.5. If we define a relation H on the objects of the category Tol by $(X,Y) \in H$ if $K(X) \cong K(Y)$ then the preceding proposition says that this relation is an equivalence. Now we can state the main theorem of this paper.

Theorem. The homotopical equivalence of tolerance spaces coincides with the relation H.

The proof will be presented as a corollary of a sequence of propositions which constitute the rest of this paper.

§ 4. Proof of the main theorem

4.1. Proposition. If Y is a contracted space of X, then $Y \sim X$.

Proof. Let $Y = X - \{x\}$; let $y \in X$ be such that $st\ x \subset st\ y$, $y \neq x$. Let $g: Y \longrightarrow X$ be the inclusion and let $f: X \longrightarrow Y$ sends the x to the y and leave the other points fixed. We claim that these mappings satisfy i) $fg \sim 1_Y$, ii) $gf \sim 1_X$.

ad i) This is clear for $fg = 1_Y$.

ad ii) Define $F: X \times I_2 \longrightarrow X$ putting $F(-,1) = gf(-)$, $F(-,2) = 1_X(-)$. We must show F to be a homomorphism or equivalently that $F(st(z,i)) \subset st\ F((z,i))$ for each $z \in X$ and $i = 1,2$. Let $z \neq x$. Then $F(z,i) = z$ and the inclusion holds. If $z = x$, $i = 2$ then $F(st(x,2)) = (\{y\} \cup st\ x) = st\ x = st\ F((x,2))$. If $z = x$, $i = 1$ then $F(st(x,1)) \subset st\ y = st\ F((x,1))$.

Corollary. All elements of a contractible sequence of a space X are homotopically equivalent.

4.2. Convention. i) For a homeomorphism $f:X \to X$ denote by X_f^i the subspace of X spanned by the fixed points of the homeomorphism f^i (the i-th iteration of f).

ii) Denote by m(X) the least common multiple of all natural numbers not exceeding a cardinal number of the space X. Let us notice that for any homomorphism of a space X into itself and $m = m(X)$ we have $f^m / X_f^m = 1_{X_f^m}$.

4.3. Lemma. Let $F_2 : f \sim g : (X,R) \to (X,R)$ be homomorphism. Then for every $(x,y) \in R$ also $(f^i(x), f^i(y)) \in R$.

Proof. The proof will be carried out by induction. For $i = 0$ the statement is obvious. Suppose we have proved the statement for $i - 1$; then $(f^{i-1}(x), g^{i-1}(y)) \in R$ and hence $(f^i(x), g^i(y)) = (F_2(f^{i-1}(x),1),\ F_2(g^{i-1}(y),2)) \in R$.

4.4. Proposition. If $F_2 : f \sim g : (X,R) \to (X,R)$ are homotopical and $m(X) = m$, then the subspace X_g^m is an element of a contractible sequence of the subspace $X_f^m \cup X_g^m$.

Proof. For an $x \in X_f^m \setminus X_g^m$ we will find a $y \in X_g^m$ such that $y = g^m(x)$. If $z \in X_f^m \cup X_g^m$ and $(x,z) \in R$ then either $z \in X_g^m$ or $z \in X_f^m$. In the first case, from $(x,z) \in R$ we obtain $(g^m(z), g^m(x)) \in R$. Thus, using the assumptions and the equality $y = g^m(x)$, we see that $(z,y) \in R$. In the second case, $(z,x) \in R$ implies $(f^m(z), g^m(x)) \in R$ and hence again $(z,y) \in R$. Thus, $\operatorname{st} x \subset \operatorname{st} y$ and the proof is completed.

4.5. Proposition. If $F_n : 1_X \sim f : X \to X$ and $m(X) = m$, then X_f^m is an element of a centralising sequence of X.

Proof. We are going to prove the proposition by induction. If $n = 1$, the proposition is evidently true. Let $n > 1$. Assume that the proposition is proved for all $m \leq n - 1$. Thus we can find homomorphisms g and F_{n-1} so that $F_{n-1}: 1_X \sim g: X \longrightarrow X$ and there is a $G_2: g \sim f$. According to the assumption and Proposition 4.4, X_g^m is an element of centralising sequences of the space X and $X_f^m \cup X_g^m$. We also know that X_f^m is an element of a centralising sequence of $X_f^m \cup X_g^m$. As $X_f^m \cup X_g^m$ is an element of a centralising sequence of the space X we can conclude that X_f^m is also an element of a centralising sequence of X. The proof is finished.

4.6. Proposition. Let X, Y be tolerance spaces and $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ homomorphisms. Then $X_{gf}^i \cong Y_{fg}^i$ for every $i > 0$.

Proof. We will show that f/X_{gf}^i is an isomorphism between X_{gf}^i and Y_{fg}^i. Let $x \in X_{gf}^i$. Then $f(x) = (fg)^i(f(x)) \in Y_{fg}^i$. That the f/X_{gf}^i is 1-1 is an obvious consequence of the fact that $z = (gf)^i(z) = g(fg)^{i-1}f(z)$ holds for each point from X_{gf}^i. To show that the map is onto consider a point $y \in Y_{fg}^i$. Then f/X_{gf}^i sends the point $g(fg)^{i-1}(y) \in X_{gf}^i$ to the very point y. We see immediately that $(f/X_{gf}^i)^{-1}$ is a homomorphism, too.

4.7. Proposition. If X and Y are homotopically equivalent then there exist centralising sequences of them such that there is an element of the centralising sequence of the space X which is isomorphic to an element of the centralising sequence of Y.

Proof. Assume $m(X) = m, m(Y) = n$. Then $X_{gf}^m = X_{gf}^{mn} \cong Y_{fg}^{mn} = Y_{fg}^n$.

4.8. The proof of the main theorem: First let us assume the spaces X, Y to have isomorphic centres, i.e., $(X,Y) \in H$. Then using the corollary of Proposition 4.1 we see $X \sim Y$. On the other hand, let $X \sim Y$. As a result of Proposition 4.7 we have $K(X) \cong K(Y)$, i.e., $(X,Y) \in H$.

R e f e r e n c e s :

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