Karel Svoboda
On surfaces in $E^3$ with constant Gauss curvature

Commentationes Mathematicae Universitatis Carolinae, Vol. 19 (1978), No. 4, 755--761

Persistent URL: http://dml.cz/dmlcz/105890

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1978

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON SURFACES IN $\mathbb{R}^3$ WITH CONSTANT GAUSS CURVATURE

Karel SVOBODA, Brno

Abstract: A global characterization of surfaces in $\mathbb{R}^3$ with constant Gauss curvature.

Key words: Surface, Gauss and mean curvatures, integral formula.

AMS: 53C45

H. Fath el Bab introduced in \cite{1} the conditions implying $H = \text{const}$ on a surface $M$ in $\mathbb{R}^3$. In what follows, we apply the method used in \cite{1} and prove an analogous theorem for the Gauss curvature $K$ of $M$.

Let $M$ be a surface in the 3-dimensional Euclidean space $\mathbb{R}^3$ and $\partial M$ its boundary. On $M$, consider fields of orthonormal frames $\{M_1, M_2, M_3\}$ with $M_1, M_2 \in T(M), T(M)$ being the tangent bundle of $M$. Then we have

\begin{align}
\text{d}M &= \omega^1 M_1 + \omega^2 M_2, \\
\text{d}M_1 &= \omega^2 M_2 + \omega^3 M_3, \\
\text{d}M_2 &= -\omega^3 M_1 + \omega^3 M_3, \\
\text{d}M_3 &= -\omega^3 M_1 - \omega^3 M_2
\end{align}

and (see \cite{2}, p. 8)
\[ (2) \quad \omega^3_1 = a \omega^1 + b \omega^2, \quad \omega^3_2 = b \omega^1 + c \omega^2; \]
\[ (3) \quad \Delta a = da - 2b \omega^1_1 = \alpha \omega^1 + \beta \omega^2, \]
\[ \Delta b = db + (a-c) \omega^1_1 = \beta \omega^1 + \gamma \omega^2, \]
\[ \Delta c = dc + 2b \omega^2_1 = \gamma \omega^1 + \delta \omega^2; \]
\[ (4) \quad \Delta \alpha = d\alpha - 3b \omega^1_1 = \Delta \omega^1 + (B-bK) \omega^2, \]
\[ \Delta \beta = d\beta + (\alpha - 2 \gamma) \omega^1_1 = (B+bK) \omega^1 + (C+aK) \omega^2, \]
\[ \Delta \gamma = d\gamma + (2\beta - \delta) \omega^2_1 = (C+cK) \omega^1 + (D+bK) \omega^2, \]
\[ \Delta \delta = d\delta + 3\gamma \omega^2_1 = (D-bK) \omega^1 + \delta \omega^2; \]

where
\[ (5) \quad K = ac - b^2 \]
is the Gauss curvature of \( M \).

The covariant derivatives \( K_{ij}, K_{ij} \) (\( i,j = 1,2 \)) of \( K \), defined by
\[ (6) \quad dK = K_1 \omega^1 + K_2 \omega^2, \]
\[ dK - K_2 \omega^1_1 = K_{11} \omega^1 + K_{12} \omega^2, \quad dK_2 + K_1 \omega^2_1 = K_{12} \omega^1 + \]
\[ + K_{22} \omega^2 \]
are given, according to (3) and (4), by
\[ (7) \quad K_1 = a \gamma - 2b \beta + c \alpha, \quad K_2 = a \delta - 2b \gamma + c \beta; \]
\[ (8) \quad K_{11} = aC - 2bB + cA + 2(\alpha \gamma - \beta^2) + (ac - 2b^2)K, \]
\[ K_{12} = aD - 2bC + cB + (a \delta - \beta \gamma) - b(a+c)K, \]
\[ K_{22} = aE - 2bD + cC + 2(\beta \delta - \gamma^2) + (ac - 2b^2)K. \]

Now, we formulate the
Theorem 1. Let $M$ be a surface in $\mathbb{R}^3$ with $K > 0$ and $\partial M$ its boundary. Let $V_1, V_2 \in T(M)$ be orthonormal vector fields on $M$ such that

$$V_1 K = 0, V_2 K = 0$$

on $\partial M$ and

$$V_1 V_1 K = 0, V_2 K = 0$$

on $M$. Then $K = \text{const}$ on $M$.

Proof. Consider a 1-form

$$\varphi = R_1 \omega^1 + R_2 \omega^2$$

on $M$. The covariant derivatives of $R_i$ $(i = 1, 2)$ being defined by

$$dR_1 - R_2 \omega_1^2 = R_{11} \omega^1 + R_{12} \omega^2,$$
$$dR_2 + R_1 \omega_1^2 = R_{21} \omega^1 + R_{22} \omega^2$$

we have, according to [1], p. 247-250, the integral formula

$$\int_{\partial M} \left[ (R_{11} R_{22} - R_{12} R_{21}) \omega^1 + (R_{1} R_{22} - R_{2} R_{12}) \omega^2 \right] =$$
$$= \int_{M} \left[ 2(R_{11} R_{22} - R_{12} R_{21}) - (R_1^2 + R_2^2) K \right] \omega^1 \wedge \omega^2.$$

Now, let us choose the tangent frames associated to $M$ in such a way that $v_1 = V_1$, $v_2 = V_2$. Then it follows from

$$V_1 K = K_1, V_2 K = K_2$$

and

$$V_1 V_1 K = K_{11} + K_2 \cdot \omega_1^2(v_1).$$

Thus we have, using (9), (10),

- 757 -
\[ K_1 = 0, K_2 = 0 \]
on \( \partial M \),
\[ K_{11} = 0, K_2 = 0 \]
on \( M \) and hence the integral formula (11), re-written for the 1-form \( K_1 \omega^1 + K_2 \omega^2 \), yields
\[ \int_M (2K_{12}^2 + K_1^2 K) \omega^1 \wedge \omega^2 = 0. \]
Thus especially
\[ K_1 = K_1 K = 0 \]
on \( M \), i.e. \( K = \text{const} \) on \( M \).

Remark that the surfaces with \( K = \text{const} \) depend on 4 functions of 1 variable.

Following [1], we are going to prove that there are, locally, surfaces \( M \) in \( E^3 \) possessing two orthonormal tangent vector fields \( V_1, V_2 \) such that \( V_2 K = 0, V_1 V_1 K = 0 \) and with \( K \) not constant on \( M \). For this purpose, we shall prove that the surfaces satisfying the preceding conditions depend on 4 functions of 1 variable.

The considered surfaces are defined by the system (4) and
\begin{align*}
(12) \quad V_2 K &= a\gamma - 2b\gamma + c\beta = 0, \\
V_1 V_1 K &= a\gamma - 2b\gamma + c\beta + 2(\alpha \gamma - \beta^2) + (ac - 2b^2) K = 0.
\end{align*}
Because of \( K = \text{const} \), we have \( V_1 K = K_1 \neq 0 \). By exterior differentiation of (4) we obtain
\begin{align*}
(13) \quad \Delta \Delta \wedge \omega^1 + \Delta B \wedge \omega^2 &= (4\beta K + bK_1) \omega^1 \wedge \omega^2, \\
\Delta B \wedge \omega^1 + \Delta C \wedge \omega^2 &= [(3\gamma - 2\alpha) K - aK_1 + bK_2] \omega^1 \wedge \omega^2,
\end{align*}

- 758 -
\[ \Delta C \wedge \omega^1 + \Delta D \wedge \omega^2 = [(2 \sigma - 3 \beta)K - bK_1 + cK_2] \omega^1 \wedge \omega^2, \]
\[ \Delta D \wedge \omega^1 + \Delta E \wedge \omega^2 = -(4 \gamma K + bK_2) \omega^1 \wedge \omega^2, \]

where

\begin{align*}
(14) \quad & \Delta A = dA - 2(2B + bK) \omega^2_1, \\
& \Delta B = dB + [A - 3C - (2a+c)K] \omega^2_1, \\
& \Delta C = dC + 2(B-D) \omega^2_1, \\
& \Delta D = dD + [3C - E + (a+2c)K] \omega^2_1, \\
& \Delta E = dE + 2(2D + bK) \omega^2_1. 
\end{align*}

Differentiating (12) and applying (3), (4), (14) we get

\begin{align*}
(15) \quad & a \Delta \sigma - 2b \Delta \gamma + c \Delta \beta + \sigma \Delta a - 2 \gamma \Delta b + \beta \Delta c - K_1 \omega^2_1 = 0, \\
& a \Delta C - 2b \Delta B + c \Delta A + \\
& + [C + c(2K - b^2)] \Delta a - 2 [B + b(3K - b^2)] \Delta b + \\
& + [A + a(2K - b^2)] \Delta c + 2(\alpha \Delta \gamma - 2\beta \Delta \beta + \gamma \Delta \alpha) + \\
& + 2K_1 \omega^2_1 = 0. 
\end{align*}

With regard to the second equation (15), the closure (13), (15) of the system (4), (12) contains \( q = 4 \) linearly independent forms and \( s_1 = 4 \) linearly independent exterior equations, so that \( s_2 = 0 \) and \( Q = 4 \). Applying the Cartan's lemma we obtain from (13)

\begin{align*}
\Delta A &= F_1 \omega^1 + F_2 \omega^2, \\
\Delta B &= (F_2 + 4\beta K + bK_1) \omega^1 + F_3 \omega^2, \\
\Delta C &= [F_3 + (3 \gamma - 2\alpha)K - aK_1 + bK_2] \omega^1 + F_4 \omega^2, \\
\Delta D &= [F_4 + (2 \sigma - 3 \beta)K - bK_1 + cK_2] \omega^1 + F_5 \omega^2, \\
\end{align*}
\[ \Delta E = (F_5 - 4rK - bK^2)\omega^1 + F_6 \omega^2, \]

the functions \( F_1, \ldots, F_6 \) satisfying two independent relations obtained from (15) by elimination of \( \omega^2 \). Thus, \( N = 4 \) and the general solution of the considered system depends on 4 functions of 1 variable.

Finally notice that the theorem 1 and that one due to H. Fath el Bab can be generalized to this form:

**Theorem 2.** Let \( M \) be a surface in \( \mathbb{R}^3 \) with \( K > 0 \) and \( \partial M \) its boundary. Let \( F(H, K) \) be a non-zero function defined on \( M \). Let \( V_1, V_2 \in T(M) \) be orthonormal vector fields such that

\[ V_1 F(H, K) = 0, \quad V_2 F(H, K) = 0 \]
on \( \partial M \) and

\[ V_1 V_1 F(H, K) = 0, \quad V_2 V_2 F(H, K) = 0 \]
on \( M \). Then \( F(H, K) \) = const on \( M \).

The proof of this assertion is analogous to the above mentioned one.

**References**


Katedra matematiky FS VUT
Gorkého 13
60200 Brno
Československo

(Oblatum 6.6. 1978)