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DEGREES OF INTERPRETABILITY

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Abstract: T is a fixed theory containing arithmetic. For sentences \( \varphi, \psi \) in the language of T, \( \varphi \leq_T \psi \) means that T with the additional axiom \( \varphi \) is relatively interpretable in T with the additional axiom \( \psi \). The structure \( V_T \) of degrees induced by \( \leq_T \) is considered and various algebraic properties of \( V_T \) are exhibited. For example, if T is essentially reflexive, then \( V_T \) is a distributive lattice with 0 and 1 and no element except 0 and 1 has a complement.

Key words: Interpretability, axiomatic theory, preorder on theories.

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Introduction. In this paper we consider formal axiomatic theories. Intuitively, some of these theories are stronger than others. This is certainly related to the question of consistency. As is well known, all the famous results concerning the consistency of the axiom of choice, continuum hypothesis and their negations were reduced to finding some interpretations. In this work we use interpretations as a mean to explicate the notion that a theory S is stronger or more complex than a theory T: it is just in the case that T is interpretable in S. In this way we have defined a (partial) preorder on theories and we may ask what properties this preorder has. In particular, is it den-
First of all, let us restrict ourselves to theories of the form \((T, \varphi)\) arising by adding one new axiom to a fixed theory \(T\). Hence we define the ordering only for sentences of \(T: \varphi \leq_T \psi \iff (T, \varphi)\) is interpretable in \((T, \psi)\). The restriction to theories of this form is convenient because we may consider only one fixed language, and it is also natural because it corresponds to the situation that we work in some theory and we are interested in the strength of additional axioms. Sentences \(\varphi\) and \(\psi\) have the same degree (notation \(\varphi \equiv_T \psi\)) if both \(\varphi \leq_T \psi\) and \(\psi \leq_T \varphi\). \(V_T\) is the set of all degrees. \(V\) is a partially ordered set with greatest and lowest element and it is a lower semilattice where meet is the disjunction of sentences.

Now there are two kinds of questions we have to solve. Firstly, questions concerning algebraic properties of the semilattice \(V_T\): are there incomparable elements in \(V_T\), is \(V_T\) a lattice?, are there complements in \(V_T\) ?, etc. Secondly, the questions on syntactical complexity: what is the simplest sentence in a given degree?

As to the first kind of questions, it follows from the results of R.G. Jeroslow [J] that for reasonable theories the ordering on \(V_T\) is dense and that there are many incomparable elements. We shall further show that for every degree \(d \neq 0, 1\) there are degrees incomparable with \(d\). If \(T\) is essentially reflexive then \(V\) is a distributive lattice. No element in \(V_T\) distinct from 0 and 1 has a complement.

If the theory \(T\) is essentially reflexive then, furthermore, in every degree in \(V_T\) there is an arithmetical \(\Pi^0_2\) and
a $\Sigma_2$ sentence. There are degrees containing neither $\Pi_1$ sentences nor $\Sigma_1$ sentences, but $\Pi_1$ sentences are in $V_T$ cofinal whereas $\Sigma_1$ sentences are not.

J. Mycielski's work [M] is motivated similarly as the present paper but the author makes no restriction on theories. In his structure every degree contains with each theory $T$ many "copies" of $T$ with different language and the l.u.b. of two degrees is simply the union of sets of representatives with disjoint languages. If the theory $T$ is essentially reflexive then $V_T$ is a substructure of Mycielski's lattice according to $\leq_T$, but I was unable to decide whether also l.u.b.'s coincide.

This paper uses the method of arithmetization described in the fundamental Feferman's paper [F]. It is a continuation of papers of R.O. Jeroslow, M. Hájková and P. Hájek. It was written under supervision of P. Hájek. I would like to thank P. Hájek for the time he spent with me during many valuable discussions and for the help with translation of the work into English.

2. Preliminaries. We shall use the logical system described in [VH 1] Chapt. I, Sect. 2. The reader may omit the following part concerning logic but he is supposed to understand the statement "the theory $T$ contains arithmetic". For example, in the set theory we may use the arithmetical operation symbols $+, \cdot, ', \bar{0}$ and form arithmetical formulas.

The language $L$ of a theory can contain variables of various sorts which are distinguished by indices ($x^i, y^j$ where $i, j$ are numbers of sorts in $L$). Every theory has one
universal sort $i$ such that for every term in $L$, $T \vdash \exists x^i (t = x^i)$. We suppose to have fixed one sort as the arithmetical sort. Variables without indices will usually be variables of the arithmetical sort.

The language of Robinson and Peano arithmetic has only the arithmetical sort and operation symbols $+, \cdot, ', \bar{0}$. For the axioms see [F].

We restrict ourselves to theories $T$ satisfying the following:

(a) $T$ has a finite language, i.e. finitely many predicates, functions and sorts (we have of course at our disposal infinitely many variables $x^i_1, x^i_2, \ldots$ of every sort $i$)

(b) $T$ has a recursively enumerable set of axioms

(c) $T$ contains Robinson arithmetic, i.e. its language has the arithmetical sort and the arithmetical operation symbols and all the axioms of Robinson arithmetic are provable in $T$

(d) $T$ is consistent.

The notion of interpretation is an obvious modification of the corresponding notion for one sorted systems.

The knowledge of Feferman's paper [F] is assumed. The predicates $Tm(n)$ (number $n$ is a term), $Fm(n)$ (n is a formula), $Prf_T(n, d)$ (n is a formula, d is a sequence of formulas and it is a proof of n in T) are primitive recursive. The predicate $Pr_T(\varphi)$ ($\varphi$ is provable in T) is recursively enumerable and the relation "($T, \varphi$) is interpretable in ($S, \psi$)" is recursively enumerable whenever $T$ is finitely axiomatizable, see Lemma 5 in [HH]. The definitions of $\Pi_m$ and $\Sigma_m$ formulas can be found e.g. in [G] and $\mathcal{PA}$-formulas are defin-
The sets $\Pi_n$ and $\Sigma_n$ are closed under conjunction, disjunction and bounded quantification; in addition, $\Pi_n$ and $\Sigma_n$ is closed under universal and existential quantification respectively. The negation of a $\Pi_n$ formula is a $\Sigma_n$ formula and vice versa. The set $PR$ is included in $\Sigma_1$ and the conjunction, disjunction, negation and bounded quantification of PR-formulas is always P-equivalent to a PR-formula, where $P$ is the Peano arithmetic. All formulas without unbounded quantifiers are PR.

The definition of numeration and binumeration are known (see [F]). A relation is primitie recursive iff it is binumerable by a PR-formula (in any theory). For every theory $T$, a relation is recursively enumerable iff it is numerable in $T$ (by a $\Sigma_1$-formula). Every finite set $A = \{a_1, \ldots, a_n\}$ has a natural PR-binumeration $x = \bar{a}_1 \lor \cdots \lor x = \bar{a}_n$ which is denoted by $[A]$.

We shall use the Feferman's formulas $T_m(x), F_m(x), St_\xi(x), Prf_\xi(x, y), Prf_\xi(x), Con_\xi$ which are real " $x$ is a (formal) term of $L"$, " $x$ is a formula", " $x$ is a sentence", " $y$ is a proof of the formula $x"$, "the formula $x$ is provable" and "the theory described by $\alpha$ is consistent". These formulas are formalizations of the related meta-mathematical notions. First four of them are PR and binumerate the sets of all terms, formulas etc., the formula $Prf_\alpha$ is $\Sigma_1$ and the formula $Con_\alpha$ is $\Pi_1$ whenever $\alpha$ is a $\Sigma_1$-formula.

Further we shall extensively use the Feferman's diagonal lemma: for every theory $T$ and for every $T$-formula $\psi(x)$ there is a sentence $\varphi$ such that $T \vdash \varphi \equiv \psi(\bar{\varphi})$. 

- 793 -
3. The semilattice of degrees of interpretability and its basic properties. In this section we shall give the basic definition and collect the most obvious facts. I include also some nontrivial results of general character.

3.1. Definition. Let $T$ be a theory, let $\varphi$, $\psi$ be sentences in the language of $T$. $\varphi$ is said to be $T$-below $\psi$ if the theory $(T, \varphi)$ is interpretable in $(T, \psi)$. This relation is denoted by $\varphi \leq_T \psi$.

3.2. Lemma. (a) $\leq_T$ is reflexive and transitive.
(b) If $T \vdash \psi \rightarrow \varphi$ then $\varphi \leq_T \psi$.

3.3. Theorem. If both $\varphi \leq_T \psi_1$ and $\varphi \leq_T \psi_2$ then $\varphi \leq_T \psi_1 \lor \psi_2$.

Proof. For simplicity, let us restrict ourselves to the case that the language of $T$ consists only of one sort and of one binary predicate $\varepsilon$. We have two interpretations $\vartheta$ and $\alpha$ of $(T, \varphi)$ in $(T, \psi_1)$ and $(T, \psi_2)$ respectively and we have to determine a new interpretation $\perp$ of $(T, \varphi)$ in $(T, \psi_1 \lor \psi_2)$. Let $\vartheta_1(x)$ be the definition of the sort $x^* \in (T, \psi_1)$, $\vartheta_2(x)$ be the definition of the sort $x^\alpha \in (T, \psi_2)$ (the ranges of interpretations $\vartheta, \alpha$). Let $E_1(x, y)$ and $E_2(x, y)$ be definitions of $x^*$ and $x^\alpha$ in $(T, \psi_1)$ and $(T, \psi_2)$ respectively. Let us define a new sort $x^\perp$ and new $\varepsilon$ (in $(T, \psi_1 \lor \psi_2)$) as follows:

$$\exists x^\perp (x = x^\perp) \equiv (\vartheta_1 \land \vartheta_2 \land \varepsilon_1(x)) \lor (\neg \vartheta_1 \land \varepsilon_2(x))$$

$$x^\perp \varepsilon_1 \psi^\perp \equiv (\vartheta_1 \land E_1(x^\perp, \psi^\perp)) \lor (\neg \vartheta_1 \land \vartheta_2 \land E_2(x^\perp, \psi^\perp)).$$

Now it is easy to check that for any formula $\chi$,

$$T, \psi_1 \vdash x^\perp = x^*$$
$$T, \neg \psi_1 \land \psi_2 \vdash x^\perp = x^\alpha$$

and that $\perp$ is indeed an interpretation of $(T, \varphi)$ in $(T, \psi_1 \lor \psi_2)$.
The last theorem shows how \( \preceq_T \) is related to the Lindenbaum algebra of sentences (with contradiction as the greatest element).

3.4. **Definition.** (a) We say that a sentence \( \varphi \) has the same degree as \( \psi \) (notation: \( \varphi \equiv_T \psi \)) iff both \( \varphi \preceq_T \psi \) and \( \psi \preceq_T \varphi \).

(b) The degree \([\varphi]\) of a sentence \( \varphi \) is the set \( \{\psi; \varphi \equiv_T \psi\} \). The set of all degrees is denoted by \( V_T \).

(c) \([\varphi] \preceq_T [\psi]\) iff \( \varphi \preceq_T \psi \).

3.5. **Lemma.** (a) \((V_T, \preceq_T)\) is a lower semilattice and \([\varphi] \wedge [\psi] = [\varphi \vee \psi]\).

(b) \(1_T = \{\varphi; \top \vdash \bot \varphi\}\) is its greatest element and \(0_T = \{\varphi; (T, \varphi)\) is interpretable in \( T\}\) is its least element.

This is a consequence of Theorem 3.3 and the fact that if \((T, \psi)\) is consistent and \( \varphi \preceq_T \psi \) then \((T, \varphi)\) is also consistent. The following lemma follows from Theorem 3.3 by elementary logic.

3.6. **Lemma.** (a) Let \( \varphi \preceq_T \psi \). Then there is a sentence \( \varphi' \) such that \( \varphi \equiv_T \varphi' \) and \( T, \psi \vdash \varphi' \).

(b) If \( \varphi \preceq_T \psi \land \varphi' \) then \( \varphi \preceq_T \psi \).

(c) If \( \varphi \preceq_T \psi \) then \([\psi \rightarrow \varphi] = 0_T \).

Proof. (a) It suffices to choose \( \varphi' = \varphi \lor \psi \) and use 3.3 and 3.2 (b).

(b) Let \( \varphi \preceq_T \psi \land \varphi' \); furthermore, we have \( \varphi \preceq_T \psi \land \varphi \). By 3.3, we have \( \varphi \preceq_T (\psi \land \varphi) \lor (\psi \land \varphi) \) and the last formula is equivalent to \( \psi \).

(c) \( \psi \rightarrow \varphi \preceq_T \psi \) by 3.2 and \( \varphi \preceq_T \psi \) by assumption. Obviously \( \psi \rightarrow \varphi \preceq_T \land \psi \), thus by 3.2 (a) and 3.3 we
have $\psi \rightarrow \varphi \iff \forall \varphi \land \neg \forall \varphi$ and the last sentence is of degree zero.

Observe that the converse of 3.6 (c) does not hold. Choose a refutable sentence for $\varphi$ and let $\psi$ be independent and such that $(T, \neg \psi)$ is interpretable in $T$. Then $[\psi]_T \vdash [\varphi]_T = T$ by 3.5 (b), moreover $T \vdash \psi \rightarrow \varphi = \neg \psi$ and the sentence $\neg \psi$ is of degree zero by 3.5 (b).

The following two theorems were stated in the Feferman's paper [F]. Recall that we assume all theories to contain Robinson arithmetic.

3.7. Theorem. Let $\tau$ be arbitrary numeration of a theory $T$ in some theory $K$. Then there is a finite subtheory $F$ of Peano arithmetic such that $T$ is interpretable in $K \cup F \upharpoonright \con^\varphi$.

3.8. Theorem. Let $K$ be a theory and let $T$ be interpretable in $S$. Then to every numeration $\sigma$ of $S$ in $K$ there is a numeration $\tau$ of $T$ in $K$ such that

$$P \vdash \con_{\sigma} \rightarrow \con_{\tau}.$$  
Moreover, $\tau$ is a $\Sigma^1_1$-formula whenever $\sigma$ is. If $T$ is finitely axiomatized we may choose $\tau \equiv \Sigma[T]$.

3.9. Definition - lemma. Let $\varphi(\chi)$ be an arithmetical formula. Then $(\varphi, \chi)$ is an abbreviation for the formula $\varphi(\chi) \lor \chi = \chi$. This formula has the following properties:

(a) $P \vdash \Phi(\chi) \& Fm(\varphi(\chi)) \rightarrow P^*_{\varphi}(\chi \rightarrow \varphi(\chi)) \equiv P^*_{(\varphi, \chi)}(\varphi(\chi))$, formalized deduction theorem

(b) If $\varphi$ (bi)numerates $T$ in $K$ then $(\varphi, \overline{\varphi})$ (bi)numerates $(T, \varphi)$ in $K$.

3.10. Definition. A theory $T$ is $\Sigma^1_1$-sound iff each $\Sigma^1_1$-sentence provable in $T$ is true (in the structure $N$ of natural numbers).
3.11. Theorem. Let $\mathcal{T} \models \mathcal{P}$ and let $\tau$ be a $\Sigma_1$-numeration of $\mathcal{T}$ in $\mathcal{T}$. Then

(a) If $\varphi$ is consistent (i.e. if $(\mathcal{T}, \varphi)$ is consistent) then $\varphi \vdash \text{Con}_{\tau}(\tau, \varphi)$.

(b) If $\mathcal{T}$ is $\Sigma_1$-sound and both $\varphi$ and $\psi$ are consistent then $\text{Con}_{\tau}(\tau, \varphi)$ and $\text{Con}_{\tau}(\tau, \psi)$ is a consistent upper bound of the set $\{\varphi, \psi\}$.

(c) $[\text{Con}_{\tau}] \neq 0_{\mathcal{T}}, [\neg \text{Con}_{\tau}] = 0_{\mathcal{T}}$.

(d) $\varphi =_{\mathcal{T}} (\varphi \land \neg \text{Con}_{\tau}(\tau, \varphi))$.

(e) If $\mathcal{T}$ is finitely axiomatized and $\varphi \vdash_{\mathcal{T}} \psi$ then $\mathcal{T} \vdash \text{Con}_{\tau}(\tau, \varphi) \rightarrow \text{Con}_{\tau}(\tau, \psi)$.

Proof. (a) By 3.9 and 3.7 $(\mathcal{T}, \varphi)$ is interpretable in a certain theory $\mathcal{T} \cup \text{Con}_{\tau}(\tau, \varphi)$ which is equivalent to $(\mathcal{T}, \text{Con}_{\tau}(\tau, \varphi))$ because $\mathcal{F} \models \mathcal{P} \models \mathcal{T}$. So we have $\varphi \vdash_{\mathcal{T}} \text{Con}_{\tau}(\tau, \varphi)$ and it remains to prove $\text{Con}_{\tau}(\tau, \varphi) \vdash_{\mathcal{T}} \varphi$. Assume $\text{Con}_{\tau}(\tau, \varphi) \vdash_{\mathcal{T}} \varphi$.

Then $\text{Con}_{\tau}(\tau, \varphi)$ is consistent because $\varphi$ is, and by 3.8 (applied to $(\tau, \overline{\varphi})$) there is a $\Sigma_1$-numeration $\sigma$ of $(\mathcal{T}, \text{Con}_{\tau}(\tau, \varphi))$ such that $\mathcal{T}$ proves $\text{Con}_{\tau}(\tau, \varphi) \rightarrow \text{Con}_{\sigma}$. This is just the situation excluded by the second Gödel's theorem (see [F]): no consistent theory $\mathcal{S} \models \mathcal{P}$ can prove the formula $\text{Con}_{\varphi}$ whenever $\varphi$ is a $\Sigma_1$-numeration of $\mathcal{S}$ in any $\mathcal{F} \models \mathcal{S}$.

(b) By (a), $\text{Con}_{\tau}(\tau, \varphi)$ and $\text{Con}_{\tau}(\tau, \psi)$ is an upper bound of $\varphi$, $\psi$. We show that $(\mathcal{T}, \text{Con}_{\tau}(\tau, \varphi)$ and $\text{Con}_{\tau}(\tau, \psi))$ is consistent. Assume the contrary. Then $\mathcal{T} \vdash \text{Pr}_{\tau}(\overline{\varphi}) \lor \text{Pr}_{\tau}(\overline{\psi})$, since the last formula is $\Sigma_1$, we have $\models \text{Pr}_{\tau}(\overline{\varphi}) \lor \text{Pr}_{\tau}(\overline{\psi})$ by $\Sigma_1$-soundness. Then $\models \text{Pr}_{\tau}(\overline{\varphi})$ or $\models \text{Pr}_{\tau}(\overline{\psi})$, for example, let $\models \text{Pr}_{\tau}(\overline{\varphi})$. Let $T_0 = \{\chi : \models \tau(\overline{\chi})\}$. Then $T_0 \vdash \varphi$ and $T_0 \models \mathcal{T}$ (since each true $\Sigma_1$-sentence is provable in $\mathcal{Q}$). Thus $T_0 \vdash \varphi$ which contradicts the assumption that $(\mathcal{T}, \varphi)$ is consistent.
(c) We know \( T \vdash \text{Con}_\varphi = \text{Con}_{(\varphi, \neg \text{Con}_\varphi)} \) see [F]. By (a) and 3.2 (b) we have

\[ \neg \text{Con}_\varphi \leq_T \text{Con}_{(\varphi, \neg \text{Con}_\varphi)} =_T \text{Con}_\varphi \]

so indeed \( 0_T \leq_T \text{Con}_\varphi \). Moreover from \( \neg \text{Con}_\varphi \leq_T \text{Con}_\varphi \)
and \( \neg \text{Con}_\varphi \leq_T \neg \text{Con}_\varphi \) we get \( \neg \text{Con}_\varphi \leq_T 0_T \) using 3.3 and 3.5 (b).

(d) is a direct application of (c) to the theory \((T, \varphi)\)
and

(e) is immediate from 3.8. —4

Theorem 3.11 (b) shows that the greatest degree \( 1_T \) is not a l.u.b. of any two smaller degrees; hence there are no "upper exact pairs". The existence of lower exact pairs is an easy consequence of the next theorem 3.12. Another consequence of Theorem 3.12 is the existence of (infinitely many) incomparable elements in \( V_T \). Theorems 3.12 and 3.13 were proved by R.G. Jeroslow in [J], the latter had to be slightly reworked for our purpose. Theorem 3.14 is my contribution to the subject.

Theorem 3.12 requires some preliminaries. Let \( B \) be the set of all propositional formulas built up from infinitely many atomic formulas \( A_1, A_2, \ldots \) by Boolean operations \( \lor, \land \) and \( \neg \). The set \( B \) can be ordered by "\( \varphi \leq_B \psi \) iff \( \varphi \) is a tautological consequence of \( \psi \)". By a natural factorization similar as in 3.4 \( B \) becomes an infinite countable atomless Boolean algebra. By a positive element of \( B \) we shall mean a (equivalence class determined by) propositional formula not containing the negation sign.

3.12. Theorem. (a) If \( T \) is a consistent theory then the countable atomless Boolean algebra can be embedded into \( V_T \).

- 798 -
More precisely, there is a one-one function $f$ from $B$ to $V_T$ preserving greatest lower bounds. In particular, for $x, y \in B$ $x \preceq_3 y$ iff $f(x) \preceq_T f(y)$.

(b) If, moreover, $T \supseteq P$ and if $\tau$ is a $\Sigma_1$-enumeration of $T$ in $T$ then $f$ maps all positive members $T$-below the formula $\text{Con}_\tau$.

For the proof see [J].

3.13. Theorem. Let a theory $T$ be essentially reflexive or finitely axiomatized. Then for every $a \preceq_T b$ there is a $c \in V_T$ such that $a \preceq_T c \preceq_T b$.

Proof. By 3.6 (a) we can choose $\varphi_1 \preceq a$, $\varphi_2 \preceq b$ such that $T \vdash \varphi_2 \rightarrow \varphi_1$. There is a finitely axiomatized theory $F \subseteq T$ such that $(F, \varphi_2)$ is not interpretable in $(T, \varphi_1)$. Indeed, if $T$ is finitely axiomatized, then we may choose $F \subseteq T$ and if $T$ is essentially reflexive then $F$ exists by Theorem 6.9 in [F] and by the reflexivity of $(T, \varphi_1)$. Recall that the set of all $(F, \lambda)$ such that $(F, \lambda)$ is interpretable in $(T, \lambda)$ is recursively enumerable. By the Feferman's diagonal lemma we can construct a self-referring sentence $\psi$ saying "if $(F, \varphi_2 \lor (\varphi_1 \land \psi))$ is interpretable in $(T, \varphi_1)$ then $(F, \varphi_2)$ is interpretable in $(T, \varphi_2 \lor (\varphi_1 \land \psi))$. Then $\chi = \varphi_2 \lor (\varphi_1 \land \psi)$ is our required formula. Obviously $\varphi_1 \preceq_T \chi \preceq_T \varphi_2$, because $\varphi_2 \vdash \chi \vdash \varphi_1$. For the proof of $\chi \preceq_T \varphi_1$ and $\varphi_2 \preceq_T \chi$ see the analogous proof in [J] Theorem 3.2. Alternatively, if the reader has [J] not at his disposal, he may extract some information from the proof of our next theorem. ~

3.14. Theorem. Let $T$ be essentially reflexive or finitely axiomatized. Let $a, b \in V_T$ be such that $a \vDash 1_T, b \vDash 0_T$. Then
there is a $c \in \mathcal{V}_T$ such that $c \not\in a$ and $b \not\in c$.

Proof. Let us choose $\gamma_1 \in a$, $\gamma_2 \in b$. By the same reason as in the proof of 3.13 there is a finitely axiomatized theory $F \subseteq T$ such that $(F, \gamma_2)$ is not interpretable in $T$. Similarly as in 3.13, there are primitive recursive relations $R_1(\varphi, n)$ and $R_2(\varphi, n)$ such that

- $R_1(\varphi, n) \lor R_2(\varphi, n)$ implies $\varphi$ is a formula
- $\exists n R_1(\varphi, n)$ iff $(F, \varphi)$ is interpretable in $(T, \gamma_1)$
- $\exists n R_2(\varphi, n)$ iff $(F, \gamma_2)$ is interpretable in $(T, \varphi)$

Let the formulas $\alpha(x, y)$ and $\beta(x, y)$ binumerate $R_1$ and $R_2$ in $Q$. Let us define a diagonal sentence $\varphi$ by

1. $Q \vdash \varphi \equiv \forall y \left( \alpha(\bar{\varphi}, y) \rightarrow \exists x \leq y \beta(\bar{\varphi}, x) \right)$

We shall prove that $\varphi$ determines the required degree $c$. We have to prove $\varphi \not\in_T \gamma_1$. We shall even prove that $(F, \varphi)$ is not interpretable in $(T, \gamma_1)$. Assume that it is interpretable by some interpretation $\kappa$. Then

$$T, \gamma_1 \not\vdash \varphi^*$$

hence

2. $T, \gamma_1 \not\vdash \forall y^* \left( \alpha^*(\bar{\varphi}, y) \rightarrow \exists x^* \leq y^* \beta^*(\bar{\varphi}, x) \right)$

and, furthermore, $R_1(\varphi, p)$ for some $p$. Let $m$ be the least such $p$. Since $\alpha$ binumerates $R$, we have

$$Q \vdash \alpha(\bar{\varphi}, m) \land \forall \omega < m \not\vdash \alpha(\bar{\varphi}, \omega)$$

Since interpretations preserve provability, we have

3. $T, \gamma_1 \not\vdash \alpha^*(\bar{\varphi}, m^*)$

From (2) and (3) we obtain

$$T, \gamma_1 \not\vdash (\exists x \leq m^* \beta(\bar{\varphi}, x))^*$$

We have proved that the sentence $\exists x \leq m \beta(\bar{\varphi}, x)$ is consistent with the theory $(F, \varphi)$, hence it is consistent with $Q$. But such a simple sentence is decided in $Q$ (according to
whether $\exists n \leq m \mathcal{R}_2(\varphi, n)$ or not). So it is decided positively, hence

(4) $\exists n \leq m \mathcal{R}_2(\varphi, n)$ and

(5) $Q \vdash \exists x \leq m \beta(\varphi, x)$.

By (4), $(F, \gamma_2)$ is interpretable in $(T, \varphi)$, but from (5) and (1) we can prove $\varphi$ in $Q$. This is a contradiction because $F$ was such that $(F, \gamma_2)$ is not interpretable in $T$. So we have proved that $(F, \varphi)$ is not interpretable in $(T, \varphi)$, hence $R_1(\varphi, n)$ does not hold for any $n$, hence for each $n$

(6) $Q \vdash \forall \alpha (\varphi, \alpha)$.

It remains to prove that $\gamma_2 \not\equiv_T \varphi$. We shall again show that even $(F, \gamma_2)$ is not interpretable in $(T, \varphi)$. If it were interpretable, i.e. if $R_2(\varphi, m)$ for some $m$, then for this $m$,

(7) $Q \vdash \beta(\varphi, m)$.

From (6) and (7) we can prove $\varphi$ in $Q$, which is impossible by the same reasons as above. 

If we choose $a = b$ in Theorem 3.14 we see that to every degree different from $0_T$ and $1_T$ there is an incomparable degree.

4. The lattice of degrees of interpretability given by an essentially reflexive theory. All results of this section concern only essentially reflexive theories. Analogous problems e.g. for finitely axiomatizable theories remain open. As is known, both Peano arithmetic and Zermelo-Fraenkel set theory is essentially reflexive.

4.1. Definition. We say that a theory $T$ is reflexive if for every $n \ T \vdash \text{Con}_{[T \ r n]} T$ is essentially reflexive if every extension of $T$ with the same language is reflexive.

- 801 -
The following lemma utilizes the fact that if \( \tau(x) \) is a binumeration of a set \( T \) in \( K \) then for every \( n \)
\[
K \vdash \tau(x) \land x \leq \overline{n} \Rightarrow [T \upharpoonright n](x),
\]
see [F], Lemma 4.14.

4.2. Lemma. Let \( T \in P \) be a recursively axiomatized theory and let \( \tau \) be arbitrary binumeration of \( T \) in \( T \). Then

(a) \( T \) is reflexive iff
\[
T \vdash \text{Con}_{T \upharpoonright \overline{n}}
\]
for each \( n \).

(b) \( T \) is essentially reflexive iff for every \( T \)-sentence \( \varphi \) and for each \( n \),
\[
T, \varphi \vdash \text{Con}_{(\tau, \varphi) \upharpoonright \overline{n}}.
\]

In the remaining part of this paper we assume that \( T \in P \), \( T \) is essentially reflexive and recursively axiomatized and \( \tau \) is a binumeration of \( T \) in \( T \).

4.3. Lemma. For arbitrary sentences \( \varphi, \psi \) \( \varphi \vdash T \psi \) iff
\[
T, \psi \vdash \text{Con}_{(\tau, \psi) \upharpoonright \overline{n}}
\]
for each \( n \).

This is a form of Orey's arithmetical compactness theorem, see [F] and [HH].

4.4. Theorem. Every pair of degrees in \( V_T \) has a l.u.b., i.e. \( V_T \) is a lattice.

Proof. Let \( a, b \) be a given pair of degrees and choose \( \varphi_1 \leq a \) and \( \varphi_2 \leq b \). By the diagonal lemma there is a sentence \( \psi \) such that
\[
(1) \ T \vdash \psi \iff \forall \varphi \ ((\text{Con}_{(\tau, \varphi) \upharpoonright \overline{n}} \rightarrow (\text{Con}_{(\tau, \varphi_1) \upharpoonright \overline{n}} \land \text{Con}_{(\tau, \varphi_2) \upharpoonright \overline{n}})).
\]
We shall prove that \( \psi \) determines the required degree, i.e. that \( [\psi] = \sup \{a, b\} \). By the essential reflexivity of \( T \) (see 4.2 (b)) we have
\[
(2) \ T, \psi \vdash \text{Con}_{(\tau, \varphi) \upharpoonright \overline{n}} \text{ for each } n.
\]
The formula \( \text{Con}_{(\tau, \varphi) \upharpoonright \overline{n}} \) is the antecedent in the formula \( \psi \);
hence from (1) and (2) we have for each \( n \)

\[ T, \forall T \phi \models \text{Con}(\forall \exists \phi) \land \text{Con}(\forall \exists \overline{\phi}) . \]

Now \( \phi_1 \models \neg \psi \) and \( \phi_2 \models \neg \psi \) by 4.3, hence \( \psi \) is an upper bound. Let \( \chi \) be arbitrary upper bound. By 3.6 (b) it suffices to prove \( \psi \models \chi \land \neg \psi \). Let \( n \) be arbitrary. As \( \chi \) is an upper bound we have (by 4.3)

(3)  
\[ T, \chi \models \text{Con}(\forall \exists \phi) \land \text{Con}(\forall \exists \overline{\phi}) . \]

Moreover, by (1),

(4)  
\[ T, \neg \psi \models \exists \psi (\text{Con}(\forall \exists \phi) \land \neg \psi \land \text{Con}(\forall \exists \overline{\phi}) \land \neg \psi ) . \]

From (3) and (4) we can prove

\[ T, \chi \land \neg \psi \models \exists \psi (\pi \leq \psi \land \text{Con}(\forall \exists \phi) \land \neg \psi ) \]

hence

\[ T, \chi \land \neg \psi \models \text{Con}(\forall \exists \phi) \land \neg \psi \]

and we get \( \psi \models \chi \land \neg \psi \) by 4.3. This completes the proof. 

From 4.3 we can prove that \( [\phi] = 0 \) iff for every \( n \) \( T \)
proves \( \text{Con}(\forall \exists \phi) \land \neg \psi \). This will be used in the proof of the following lemma.

4.5. Lemma. For every theory \( T \), there is a sentence \( \psi \) such that \( [\psi] = [\neg \psi] = 0 \).

Proof. Let \( \text{neg}(x, z) \) be a formula that functionally bi-numerates negation in \( Q \), i.e. for arbitrary formula \( \phi \),

(1)  
\[ Q \models \neg \text{neg}(\overline{\phi}, x) \equiv x = \overline{\text{neg}(\phi)} . \]

Let us define a diagonal sentence \( \psi \) by

\[ T \models \psi \equiv \forall \psi (\text{Con}(\forall \exists \phi) \land \neg \psi \rightarrow \forall x (\text{neg}(\overline{\phi}, x) \rightarrow \text{Con}(\forall \exists \phi) \land \neg \psi)) . \]

By (1) we have

(2)  
\[ T \models \psi \equiv \forall \psi \left( \text{Con}(\forall \exists \phi) \land \neg \psi \rightarrow \text{Con}(\forall \exists \overline{\phi}) \land \neg \psi \right) . \]

By the reflexivity of the theory \( (T, \phi) \) we have

(3)  
\[ T, \psi \models \text{Con}(\forall \exists \phi) \land \neg \psi \]  for each \( n \).

From (2) we get
(4) \( T, \varphi \vdash \text{Con}((e, \varphi)) \upharpoonright \omega \) for each \( n \).

By the reflexivity of \((T, \text{true})\) we have

(5) \( T, \text{true} \vdash \text{Con}((e, \text{true})) \upharpoonright \omega \).

By (4) and (5)

\[ T \vdash \text{Con}((e, \varphi)) \upharpoonright \omega \]

and indeed \([\varphi] = 0_T\). Furthermore, by (2) we have

(6) \( T, \varphi \vdash \exists \psi \left( \text{Con}((e, \psi)_\omega) \land \neg \text{Con}((e, \psi)) \right) \).

From (5) and (6) (using the fact that \( \varphi \land \neg \varphi \)) we get

(7) \( T, \varphi \vdash \text{Con}((e, \varphi)) \upharpoonright \omega \).

And again by (3) and (7)

\[ T \vdash \text{Con}((e, \varphi)) \upharpoonright \omega \]

for each \( n \), i.e. \([\varphi] = 0_T\).

If we apply Lemma 4.5 to the theory \((T, \varphi)\) we get the following

**Corollary.** In every degree \([\varphi]\) there are mutually contradictory sentences of the form \( \varphi \land \psi \) and \( \varphi \land \neg \psi \).

4.6. Lemma. For arbitrary sentences \( \varphi, \psi \)

\[ T \vdash \text{Con}((e, \varphi \land \psi)) = \text{Con}((e, \varphi)) \lor \text{Con}((e, \psi)) \).

Proof. We know that for arbitrary sentences \( \chi, \chi_1, \chi_2 \),

\[ P \vdash \text{Con}((e, \varphi)) = \text{Con}((e, \chi_1 \land \chi_2)) \land P \vdash \text{Con}((e, \varphi)) = \text{Con}((e, \chi_1 \lor \chi_2)). \]

Lemma 4.6 is an easy consequence of these facts. —

Having Theorem 4.4 in mind we can use in \( V_T \) the lattice operations \( \lor \) (least upper bound, join) and \( \land \) (meet). Recall that if \( a, b \in V_T \) and \( \varphi \in \sigma \), \( \psi \in \sigma \) then \( \varphi \lor \psi \in \sigma \land \sigma \) (see 3.5 (a)) and \([\varphi \land \psi] \geq_T a \lor b\) by 3.2 (b).

4.7. Theorem. The lattice \( V_T \) is distributive.

Proof. It suffices to prove that \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \) because the dual distributivity law follows from this one.

- 804 -
Moreover, the inequality $\leq$ holds automatically in every lattice. Let us prove $\geq$. Choose $\varphi_1 \leq \alpha$, $\varphi_2 \leq \beta$, $\varphi_3 \leq \gamma$ and define diagonal formulas

$$\psi_1 = \forall \delta \left( \text{Con}(\varphi_1, \varphi_2) \land \text{Con}(\varphi_2, \varphi_3) \land \text{Con}(\varphi_3, \varphi_1) \right)$$

$$\psi_2 = \forall \delta \left( \text{Con}(\varphi_2, \varphi_1) \land \text{Con}(\varphi_1, \varphi_3) \land \text{Con}(\varphi_3, \varphi_2) \right)$$

$$\psi_3 = \forall \delta \left( \text{Con}(\varphi_3, \varphi_2) \land \text{Con}(\varphi_2, \varphi_1) \land \text{Con}(\varphi_1, \varphi_3) \right)$$

By 3.5 (a) and 4.4 we have $\varphi_2 \lor \varphi_3 \leq \beta \land \gamma$, $\varphi_1 \leq \alpha \lor \beta$, $\varphi_2 \leq \alpha \lor \gamma$, $\chi \leq \alpha \lor (\beta \land \gamma)$ and $\varphi_1 \lor \varphi_2 \leq (\alpha \lor \beta) \land (\alpha \lor \gamma)$. We have to prove that

$$\psi_1 \lor \psi_2 \leq_T \varphi_1 \lor \varphi_2 \leq_T \chi$$

By 3.6 (b) it suffices to prove

$$\psi_1 \lor \psi_2 \leq_T \chi \land \lnot \psi_1 \land \lnot \psi_2$$

By 4.3 it suffices to prove that, for each $n$,

$$T, \varphi_1, \lnot \psi_1, \lnot \psi_2 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \lnot \psi_1 \lor \varphi_2$$

We shall prove

$$T, \varphi_1, \lnot \psi_1, \lnot \psi_2 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

and use Lemma 4.6. Let $n$ be given. By the reflexivity of $(T, \varphi_1)$ we have

$$T, \varphi_1 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

By this and by the definition of $\varphi_1$ we have (using Lemma 4.6)

$$T, \varphi_1 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

hence

$$T, \varphi_1 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

From the definition of $\psi_1$, $\psi_2$ we get

$$T, \lnot \psi_1, \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

$$T, \lnot \psi_2, \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$

Putting this together we indeed have

$$T, \varphi_1, \lnot \psi_1, \lnot \psi_2 \vdash \text{Con}(\varphi_1, \varphi_2) \vdash \varphi_1 \lor \varphi_2$$
5. **Simplest sentences in a degree.** The sentence \( \psi \) produced in the theorem 4.4 was an arithmetical sentence. If we take in the theorem 4.4 the same sentence for \( \varphi_1 \) and \( \varphi_2 \) we see that in every degree in \( \mathcal{V}_T \) there is an arithmetical and syntactically simple sentence. This contrasts with the fact that in the Lindenbaum algebra e.g. of ZF there are degrees of arbitrarily high arithmetical complexities and that there are also non-arithmetical degrees, i.e. there are set sentences non-equivalent to any arithmetical sentence. In this section we shall further try to determine for some concrete formulas their position in the lattice \( \mathcal{V}_T \).

5.1. **Theorem.** If \( \text{T2P} \) is essentially reflexive and recursively axiomatized then

(a) In every degree in \( \mathcal{V}_T \) there are \( \Pi_2 \) sentences.

(b) In every degree in \( \mathcal{V}_T \) there are \( \Sigma_2 \) sentences.

**Proof.** (a) Let a degree \([\varphi]\) be given and let \( \mathcal{V} \) be a \( \Sigma_1 \) -binumeration of \( T \) in \( T \). Let us define a diagonal sentence \( \psi \) by

\[
T \vdash \psi = \forall \varphi \left( \text{Con}(\varphi, \overline{\varphi}) \rightarrow \text{Con}(\varphi, \overline{\varphi}) \land \varphi \right).
\]

The formula \( \psi \) is \( \Pi_2 \) and the proof that \( \psi \vdash \varphi \) is analogous to the proof of the theorem 4.4.

(b) Let \( \varphi, \psi \) be as above and let us take a sentence

\[
\delta \equiv \exists \varphi \left( \text{Con}(\varphi, \overline{\varphi}) \land \forall \text{Con}(\varphi, \overline{\varphi}) \right).
\]

Obviously \( \delta \) is a \( \Sigma_2 \) sentence and \( T \vdash \delta \rightarrow \psi \). So we have to prove \( \delta \vdash \psi \). By 3.11 (d) \( \forall \varphi \exists \psi \land \text{Con}(\varphi, \overline{\varphi}) \). Furthermore, we have

\[
T \vdash \exists \varphi \text{Con}(\varphi, \overline{\varphi}) \rightarrow \exists \psi \text{Con}(\varphi, \overline{\varphi}) \land \varphi,
\]

\[
T, \varphi \land \exists \psi \text{Con}(\varphi, \overline{\varphi}) \vdash \delta
\]

and hence \( \delta \vdash \psi \).
5.2. **Theorem.** Let $T$ and $S$ be theories containing Peano arithmetic, let the induction for all $T$-formulas be provable in $T$ and let $T$ enable the coding of finite $n$-tuples of $T$-objects. Then to every interpretation $*\kappa$ of $S$ in $T$ there is a $T$-formula $\varphi(x, x^*)$ such that

(a) $T \vdash \forall x \exists! x^* \varphi(x, x^*)$

(b) $T \vdash \varphi(x_1, x^*) \land \varphi(x_2, x^*) \rightarrow x_1 = x_2$

(c) $T \vdash \varphi(x, x^*) \land \forall x^* x^* \rightarrow \exists y \varphi(x, y^*)$

(d) for every arithmetical $\Sigma_1$-formula $\varphi(x, \ldots)$

$T \vdash \varphi(x, x^*) \land \cdots \rightarrow (\varphi(x, \ldots) \rightarrow \varphi^*(x^*, \ldots))$

For the proof see e.g. [H].

If we apply Theorem 5.2 to a $\Sigma_1$-sentence $\varphi$ we get $T \vdash \varphi \rightarrow \varphi^*$. The dual statement for $\Pi_1$-sentence $\sigma$ claims $T \vdash \sigma^* \rightarrow \sigma$. This fact has important consequences.

5.3. **Corollary.** Let $T$ have the properties required in Theorem 5.2. If $\psi$ is a $T$-sentence and $\varphi$ is a $\Pi_1$-sentence then $\varphi \leq_T \psi$ implies $T, \psi \vdash \varphi$.

5.4. **Corollary.** Let $T$ have the properties from Theorem 5.2 and let $\varphi_1, \varphi_2$ be $\Pi_1$-sentences. Then

$[\varphi_1 \land \varphi_2] = [\varphi_1] \lor [\varphi_2]$.

The following definition 5.5 and lemma 5.6 show the connection that interpretability has to partially conservative sentences (studied by D. Guaspari).

5.5. **Definition** [G]. A sentence $\varphi$ is said to be $\Pi_1$-conservative over $T$ if for every $\Pi_1$-sentence $\sigma$, $T, \varphi \vdash \sigma$ implies $T \vdash \sigma$.

5.6. **Lemma** [G]. Let $T$ be reflexive and satisfy the assumptions of 5.2. Then $\varphi$ is $\Pi_1$-conservative iff $[\varphi] = \emptyset$.

Proof. $T$ is essentially reflexive hence $T, \varphi \vdash \text{Con}_T(\tau, \varphi) \land \varphi$.
for each n. The sentence $\text{Con} \ldots$ is $\Pi_1$ hence by the $\Pi_1$-conservativity of $\varphi$ we have $T \vdash \text{Con}_{(\tau, \varphi)}^{\tau_n}$ and by Lemma 4.3 indeed $[\varphi] = 0_T$.

Assume conversely $[\varphi] = 0_T$. Let $T, \varphi \vdash \pi$ and $\pi \in \Pi_1$. We have to prove $T \vdash \pi$. Let $\star$ be an interpretation of $(T, \varphi)$ in $T$. Then $T, \varphi \vdash \pi$ implies $T \vdash \pi^\star$. By Theorem 5.2 or Corollary 5.3 we have $T \vdash \pi^\star$.

5.7. Rosser's sentences. In the rest of the paper assume that $T$ is P or ZF and $\tau$ is a PR-binumeration of $T$ in $T$. Let us define sentences $\varphi$ and $\pi$ (the former using the diagonal lemma):

$$\varphi = \forall x \exists \bar{y} \;
\text{Pr}_{\tau}^\star (-\bar{y}, \bar{y}) \rightarrow \exists x \leq \bar{y} \; \text{Pr}_{\tau}^\star (\overline{\tau}, x)$$

$$\pi = \forall x \exists \bar{y} \;
\text{Pr}_{\tau}^\star (-\bar{y}, \bar{y}) \rightarrow \exists y < x \; \text{Pr}_{\tau}^\star (\overline{\tau}, \bar{y}) .$$

To be more exact $\varphi$ is defined using the formula $\neg (x, z)$ similarly as in 4.5. The sentences $\varphi$ and $\pi$ have the following properties

(a) $[\varphi] = [-\pi] \not\in 0_T$, $[\neg \varphi] = [\pi] \not\in 0_T$

(b) $[0_T] = [\varphi] \land [\pi]$

(c) $[\text{Con}_{\tau}] = [\varphi] \lor [\pi]$

(d) $[\varphi] < [\text{Con}_{\tau}]$, $[\pi] < [\text{Con}_{\tau}]$

Proof. It is well known that

(i) The sentence $\varphi$ is independent on $T$. The proof can be formalized in $(T, \text{Con}_{\tau})$ and since $T \vdash \neg \varphi \rightarrow \pi$ we have

(ii) $T \vdash \text{Con}_{\tau} \rightarrow \text{Con}_{\tau, \varphi}$, $T \vdash \text{Con}_{\tau} \rightarrow \text{Con}_{\tau, \pi}$.

(iii) $T \vdash \text{Con}_{\tau} = \varphi \land \pi$. By Corollary 5.4 we have $[\text{Con}_{\tau}] = [\varphi] \lor [\pi]$.

(iv) $T \vdash \varphi \rightarrow \text{Con}_{\tau}$, $T \vdash \pi \rightarrow \text{Con}_{\tau}$; otherwise we would reach a contradiction with the second Gödel's theorem (using (ii)).

- 808 -
(v) \( T \vdash \sigma \)
otherwise we would have \( T \vdash \phi = \text{Con}_x \) (by (iii)) which con­
tradicts (iv).

(vi) \([\phi] \neq 0_T, [\sigma] \neq 0_T\)
since \( \phi \) and \( \sigma \) are unprovable \( \Pi_1 \)-sentences, see 5.3.

(vii) \( \neg \phi \leq_T \sigma, \neg \sigma \leq_T \phi \)
since by 3.11 (d), we have \( \sigma \land \neg \text{Con}_{(x, [\neg \phi])} \leq_T \sigma \) and, by (ii),
we have \( \sigma \land \neg \text{Con}_x \leq_T \sigma \land \neg \text{Con}_{(x, [\neg \phi])} \).
In \( T, \sigma \land \neg \text{Con}_x \) implies \( \neg \phi \) by (iii). The proof of
\( \neg \sigma \leq_T \phi \) is similar. Now it is clear that \([\phi] = [\neg \sigma]\) and
\([\neg \phi] = [\sigma]\) since \( T \vdash \neg \phi \rightarrow \sigma \).

(viii) The property (d) follows from (a), (b), (c). This
completes the proof. 

Let us point out that 5.7 (a) shows that a degree diffe­
rent from \( 0_T, 1_T \) can contain both \( \Pi_1 \) and \( \Sigma_1 \) sentence.

5.8. The negation of the Rosser's sentence informally
says "there is a proof of my negation such that no my proof is
less or equal". Let us slightly change this sentence and de­
fine
\[ \sigma = \exists x (\exists f_x (\exists \phi_x, x) \land \forall \psi \neq x \neg \exists f_x \neg \text{Con}_{(x, \psi)}).\]
This sentence has the following properties

(a) \( \sigma \leq_T \text{Con}_{(x, \neg \text{Con}_x)} \)

(b) \( \sigma \leq_T \text{Con}_x \).

Proof. (i) If \( T \vdash \neg \sigma \) then \( T \vdash \neg \text{Con}_x \). By the for­
malization of this fact we have

(ii) \( T \vdash \text{Con}_{(x, \neg \text{Con}_x)} \rightarrow \text{Con}_{(x, \phi)} \),
and by 3.11 (a) we have \( \sigma \leq_T \text{Con}_{(x, \neg \text{Con}_x)} \).

(iii) \( T \vdash \text{Con}_x \rightarrow \neg \phi \)
since by Theorem 5.5 in [F] we have \( T, \neg \sigma \vdash \neg f_x (\neg \phi) \) and by

- 809 -
the definition of $\sigma$ we have $T, \sigma \vdash \text{Pr}_\sigma(\neg \text{Con}_\sigma)$, which implies $T, \sigma \vdash \neg \text{Con}_\sigma$.

(iv) $\sigma \not\vdash \text{Con}_\sigma$.

Assume $\sigma \not\vdash \text{Con}_\sigma$. Let $*$ be an interpretation of $(T, \sigma)$ in $(T, \text{Con}_\sigma)$. The theory $(T, \text{Con}_\sigma)$ is consistent and it remains consistent after adding the axiom of formal inconsistency. Thus it will be sufficient to find a contradiction in the theory $(T, \text{Con}_\sigma, \text{Pr}_\sigma(\neg \text{Con}_\sigma))$. Let us work in the last theory informally. Let $y$ be least such that $\text{Pr}_\sigma(\neg \text{Con}_\sigma, y)$. The formula $\text{Pr}_\sigma ...$ is in $P$, hence it is $\Sigma_1$ and by Theorem 5.2 we have $\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, y^*)$, where $y^*$ is such that $\varphi(y, y^*)$. We know that $\sigma^*$, hence

$\exists x^*(\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, x^*) \land \forall y^* (x^* \land \text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, y^*))$.

Every such $x^*$ must be $\leq y^*$ and by 5.2 (c) there is an $x$ such that $\varphi(x, x^*)$. By 5.2 (d) $\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, x^*)$ implies $\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, x)$, since $\text{Pr}_\sigma ...$ is a $\Sigma_1$-formula in $P$. By (iii) there is a $y' \neq x$ such that $\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, y')$ and for this $y'$ we have $y' \leq y$.

But $y$ was least such that $\text{Pr}_\sigma^*(\neg \text{Con}_\sigma^*, y)$. This is a contradiction.

5.9. A truth definition for a theory $T$ is a $T$-formula $\psi(x)$ such that for every $T$-sentence $\varphi$, $T \vdash \varphi \iff \psi(\varphi)$. As is known, no consistent theory has such a truth definition.

On the other hand, the Peano arithmetic has partial truth definitions. More precisely, for every $n$ there is a $\Sigma_n$-formula $\text{Tr}_n(x)$ such that for every $\Sigma_n$-sentence $\varphi$, $T \vdash \varphi \iff \text{Tr}_n(\varphi)$.

Let us define the sentences $\omega_n$ using the formulas $\text{Tr}_n(x)$ and the natural binumeration $\pi$ of axioms of the Peano arithmetic:

$\omega_n \equiv \forall x (\text{ST}_{\Sigma_n}(x) \& \text{Tr}_n(x) \rightarrow \text{Con}_{\omega_n}(x))$

("every $\Sigma_n$-true $\Sigma_n$-sentence is consistent with $\pi$ ").

These sentences have the following properties:
(a) $\omega_n \in \Pi_n$

(b) If $\sigma$ is a $\Sigma_n$-sentence then

$P, \omega_n, \sigma \vdash \text{Cn}(\sigma, \bar{\mathcal{F}})$

(c) If $\sigma$ is a $\Sigma_n$-sentence then

$P, \omega_n \vdash \sigma$ implies $P, \omega_n \vdash \text{Cn}(\sigma, \bar{\mathcal{F}})$.

(d) There is no $\Sigma_n$-sentence $\sigma$ such that $P, \sigma \vdash \omega_n$.

(e) $P \vdash \omega_1 = \text{Cn}_{\omega_1}$.

(f) Each $\omega_n$ is consistent with $P$.

Proof. (a) is obvious, (b) follows from the definition and from the fact that $P \vdash \sigma \equiv \Pi_n(\sigma)$. (d) Assume $P, \sigma \vdash \omega_n$. Then, by (b), $P, \sigma \vdash \text{Cn}(\sigma, \bar{\mathcal{F}})$ which contradicts the second Gödel’s theorem. (e) The interesting direction is $\text{Con}_{\omega_1} \rightarrow \omega_1$. It is a consequence of the fact that $P \vdash \mathcal{G}(\pi, x) \& \Pi_n(x) \rightarrow \text{Cn}(\pi, x)$ which is a generalization of the Feferman’s theorem 5.5 and is proved by induction on complexity of formulas (in $P$). (f) It is sufficient to prove $\text{ZF} \vdash \omega_n$ for each $n$.

Let us work in $\text{ZF}$ informally. Let $\mathcal{N}$ be the structure of natural numbers. $\mathcal{N}$ is known to be a model of the set $\{ x ; \sigma(x) \}$. By induction on complexity of formulas we can prove (all in $\text{ZF}$) that $\mathcal{G}(\pi, x) \rightarrow (\Pi_n(x) \equiv \mathcal{N} \models x)$. We see that every $\Sigma_n$-true $\Sigma_n$-sentence $\sigma$ holds in $\mathcal{N}$, hence $\mathcal{N} \models \langle \pi, x \rangle$, hence $\text{Cn}(\pi, x) \vdash \sigma$.

We see that every $\omega_n$ is a $\Pi_n$-sentence which is not $\Sigma_n$ in $P$. The $\omega_1$ and $\omega_2$ have analogous properties also in $\text{ZF}$:

5.10. Theorem. (a) There is no $\Sigma_1$-sentence $\sigma$ such that $\omega_1 \not\models \sigma$. In particular, the degree $[\text{Cn}_{\omega_1}]$ contains no $\Sigma_1$-sentence.

(b) The degree $[\omega_2]$ contains no $\Pi_1$-sentence.
Proof. These are consequences of 5.3 and 5.9 (d). In (a) use the fact that $\omega_1 \in \Pi_1^1$ and in (b) that $\Pi_1^1 \in \Sigma_2^1$.

Now our picture is almost complete. Every degree contains $\Pi_2^1$ and $\Sigma_2^1$-sentences. By 5.10 (b) not every degree contains $\Pi_1^1$-sentences, but by 3.11 (a), (b), $\Pi_1^1$-sentences are cofinal in $\mathcal{V}_T$. On the other hand $\Sigma_1^1$-sentences are not cofinal in $\mathcal{V}_p$ (by 5.10 (a)) and this can be generalized also for $\mathcal{V}_{ZF}$. By 5.8 it is not true that every $\Sigma_1^1$-sentence is $T$-below the sentence $\text{Con}_T$. A degree containing a $\Pi_1^1$-sentence may contain a $\Sigma_1^1$-sentence (see 5.7) or may not (see 5.10 (a)).

6. Problems. The only question concerning simple formulas in a degree reads: must a degree containing a $\Sigma_1^1$-sentence contain also a $\Pi_1^1$-sentence?

We close this paper by collecting some further open problems. The most important question we have left open reads:

Is $\mathcal{V}_T$ a lattice for finitely axiomatizable $T$? In particular, is $\mathcal{V}_{\text{GB}}$ a lattice? As a consequence of the proof of the theorem 3.4.1 in [VHZ] we have the following fact: If $\Phi(x)$ is the natural binumeration of $ZF$ and $ZF \vdash \forall x (\text{Con}_x(x, \Phi(x)) \rightarrow \text{Con}_x(x, \Phi(x)))$ then $\Phi \in \text{GB}$. It follows that the sentence produced in 4.4 is an upper bound also in $\mathcal{V}_{\text{GB}}$. Other open problems are: is every $c \in \mathcal{V}_T$, $c \uparrow T$ a l.u.b. of two smaller degrees?, is every $a \uparrow \Omega_1$, $\Omega_1$ one member of a lower exact pair?

References


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