Commentationes Mathematicae Universitatis Carolinae

Ryszard Frankiewicz; Władysław Kulpa
On order topology of spaces having uniform linearly ordered bases

Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 1, 37--41

Persistent URL: http://dml.cz/dmlcz/105900

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1979

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20, 1 (1979)

ON ORDER TOPOLOGY OF SPACES HAVING UNIFORM LINEARLY ORDERED BASES

R. FRANKIEWICZ, W. KULPA

Abstract: It is shown that a dense in itself topological space X which has a uniformity with a linearly ordered (with respect to star-refinements) base of uncountable cofinality is an ordered topological space.

Key words: Order topology, linearly ordered base of uniformity.

AMS: 54F05, 54E15

A class of topological spaces which have uniformities with linearly ordered bases (shortly, with uniform l.o. bases) contains all metrizable spaces. The topology of a metrizable space is induced by a uniformity with a countable base linearly ordered (with respect to the star-refinements of coverings). Herrlich [1] has proved (and Lynn [3] for separable metric spaces) that for each metric space X with dim X = = 0, the topology of X is induced by a linear order. Our result can be treated as an extension of the results of Herrlich and Lynn. If a space X has a uniformity with l.o. bases then X is metrizable or X is paracompact, dim X = 0, and X is a dense subspace of the limit of an inverse system over well-ordered set of discrete spaces [2]. Consequently, if X is dense in itself, then the topology of X is an order topology.

If a space X with a uniform l.o. base of uncountable cofinality has "many" isolated points then we do not know if it is true that the topology of X is an order topology. We can apply proof that such a space is a GO-space, i.e. a subspace of an order space. The special case, every topological group with linearly ordered base of neighborhoods of the neutral element is orderable, was proved in [4].

<u>Lemma 1</u> [2]. If a space X has a uniform l.o. base B of an uncountable cofinality, of $B > \aleph_0$, then for each family $\mathcal{R}$ of open sets with card $\mathcal{R} <$ of B , the intersection $\cap \mathcal{R}$ is an open set.

Proof. Let $x \in \cap \mathcal{R}$. For each $G \in \mathcal{R}$ let us choose a $P_G \in B$ such that $st(x,P_G) \subset G$. Since card $\{P_G : G \in \mathcal{R}\} <$ of B, there exists $P \in B$ such that $P \succ P_G$ ($P \succ Q$ means that P is a refinement of Q) for each $G \in \mathcal{R}$. Hence $st(x,P) \subset \cap \mathcal{R}$. Thus $\cap \mathcal{R}$ is an open set.

From Lemma 1 it follows that if a space X has a uniform l.o. base B with cf $B > \aleph_0$, then each $\mathcal{G}_\delta$ subset is open in X, consequently, dim X = 0 [2]. Indeed, let $\{V_i : i = 1,\ldots,k\}$ be a finite functionally open covering of the space X. There exists a functionally closed covering $\{F_i : i = 1,\ldots,k\}$ such that $F_i \subset V_i$, $i = 1,\ldots,k$. Each F_i is a $\mathcal{G}_\delta$ set, so it is clopen set. Put $U_1 = F_1$ and $U_j = F_j - \bigcup\{U_i : i < j\}$. The family $\{U_i : i = 1,\ldots,k\}$ is an open covering of X, $U_i \subset\subset V_i$, $U_i \cap U_j = \emptyset$ for i j, $i,j = 1,\ldots,k$. Thus dim X = 0.

<u>Lemma 2</u> [2]. Each topological space which has a uniform l.o. base is paracompact.

Proof. Since each linearly ordered set contains a cofinal and well-ordered subset we may assume that $B = \{P_\alpha : \alpha < \gamma\}$, $P_\alpha \succ_* P_\beta$ iff $\alpha > \beta$, is a well-ordered with respect to the star-refinements uniform base for X. Let **P** be an open covering of X. Define $Q = \{st(x,P_{\alpha+2}) : st(x,P_\alpha) \subset u,\ u \in \mathbf{P},\ x \in X\}$. The covering Q is a star-refinement of **P**. This implies that X is a paracompact space.

Lemma 3 [2]. If a space X has a uniform l.o. base with cf $B > \aleph_0$, then it has a uniform l.o. base B' consisting of open coverings of order 1.

Proof. Let $B = \{P_\alpha : \alpha < \gamma\}$, γ = cf B, be a well-ordered uniform base on X. Define zero-dimensional base $B' = \{Q_\alpha : \alpha < \gamma\}$. Since dim X = 0 and X is paracompact, there exists an open covering $Q_1 \succ_* P_1$ and Q_1 is of order 1. Let us assume that Q_α, $\alpha < \beta < \gamma$, are defined. By Lemma 1, there exists an open covering P such that $P \succ_* P_\beta$ and $P \succ_* Q_\alpha$, $\alpha < \beta$. Let $Q_\beta \succ P$ be an open covering of order 1.

Theorem. If a dense in itself space **X** has a uniform l.o. base of uncountable cofinality, then there exists a linear order on X inducing the topology of the space X.

Proof. Notice that T is an infinite set, then there is a linear order $<$ on T such that each $x \in T$ has elements x - 1 and m + 1 in a sense of the discrete order $<$. Indeed, let $\prec$ be an arbitrary linear order on T, then the lexicographic order on $T \times Z$, where Z is the set of integers is a discrete order. Since card T = card $(T \times Z)$, hence T has a discrete order without the first and the last element.

Let $B = \{P_\alpha : \alpha < \gamma\}$, γ = cf B, $P_\alpha \succ_* P_\beta$ iff $\alpha > \beta$, be a uniform well-ordered base consisting of open coverings of order 1. For each $x \in X$ put $x(\alpha) = u \in P_\alpha$, such that $x \in u$, and for each $u \in P_\alpha$ let $\pi(u) = \{v \in P_{\alpha+1} : v \in u\}$. Since X has bo isolated point, without loss of generality we may assume that for each $u \in P_\alpha$, card $\pi(u) \geq \aleph_0$.

Now, assume that for each $u \in P_\alpha$, $\alpha < \gamma$, it is chosen a discrete order $<$ (without the first and the last element) on $\pi(u)$ and let us assume that it is given a discrete order $<$ on each P_β, where $\beta < \gamma$ is a limit ordinal.

Define a linear order on X. For each $x, y \in X$ let us put $x < y$ iff $x(\alpha) < y(\alpha)$, where $\alpha = \min\{\beta < \gamma : x(\beta) \neq y(\beta)\}$.

Now, we shall show that the topology induced by the order $<$ is equal to the topology of the space X. Notice that $B^* = \bigcup B$ is a base for the topology of X. Let $z \in u \in P_\alpha$, $\alpha < \gamma$. There exist $z(\alpha+1) - 1$, $z(\alpha+1) + 1 \in \pi(u)$. Choose $x, y \in X$ such that $x(\alpha+1) = z(\alpha+1) - 1$, $y(\alpha+1) = z(\alpha+1) + 1$. Notice that $\langle x,y \rangle \subset u$. Now, consider an interval $\langle x,y \rangle$ and $z \in \langle x,y \rangle$. There is the least α, $\beta < \gamma$ such that $x(\alpha) \in z(\alpha)$ and $z(\beta) < y(\beta)$. If $\alpha \leq \beta$, then $z(\beta) \subset \langle x,y \rangle$. If $\beta \leq \alpha$, then $z(\alpha) \subset \langle x,y \rangle$. But $z(\alpha)$, $z(\beta)$ are open neighbourhoods of the point z. Thus the topologies are equal.

References

[1] H. HERRLICH: Ordnundsfähigkeit total-diskontinuierlicher Räume, Math. Ann. 159(1965), 77-80.

[2] A. KUCIA, W. KULPA: Spaces having uniformities with linearly ordered base, Prace Nauk. Uniwersytetu Śląskiego, Prace Mat. 3(1973), 45-50.

[3] I.L. LYNN: Linearly orderable spaces, Proc. Amer. Math. Soc. 13(1963), 454-456.

[4] P.J. NYIKOS, H.-C. REICHEL: Topologically orderable groups, Gen. Top. Appl. 5(1975), 195-204.

Uniwersytet Śląski

Instytut Matematyki

ul. Bankowa 14, Katowice

POLSKA

(Oblatum 14.3.1978)