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ON THE REGULARITY OF WEAK SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS VIA LIOUVILLE'S TYPE PROPERTY

M. GIAQUINTA, J. NECAS

Abstract: Let \( u \) be a weak solution with bounded gradient of a nonlinear elliptic system. In the present paper it is proved that the first derivatives of \( u \) are Hölder-continuous if the system satisfies a Liouville's type condition. This condition, roughly speaking, means that every solution defined on the whole \( \mathbb{R}^n \) to the system and with bounded gradient is a polynomial of at most first degree.

Key words: Regularity, weak solution, nonlinear elliptic system, Liouville's property, Sobolev space.

AMS: 35J60

§ 1. Introduction. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be a bounded domain and let us consider a nonlinear elliptic system

\[
- \frac{\partial}{\partial x_i} \left[ a_1^r(x,u,\nabla u) \right] + a_r^r(x,u,\nabla u) = - \frac{\partial x_i^r}{\partial x_i} + f^r, \quad r = 1,2,\ldots,m, \]

where \( u \in [W^{1,\infty}(\Omega)]^m, \) \( \nabla u \) is the set of the derivatives \( \frac{\partial u}{\partial x_j}, a_1^r(x,\xi,\eta), \frac{\partial a_1^r}{\partial x_j}, \frac{\partial a^r}{\partial x_j}, \frac{\partial a^r}{\partial \eta}, \)

are continuous functions on \( \Omega \times \mathbb{R}^m \times \mathbb{R}^m, \) \( f^r \in W^{1,p}(\Omega), \)

\( f^r \in W^{1,\frac{2p}{p+1}}(\Omega), \) \( p > n, \)

\[ (1.2) \quad \frac{\partial a^r}{\partial \eta_j} (x,\xi,\eta) \eta_i \eta_j r > 0 \text{ for } \eta \neq 0, \]

and the summation convention is used.

Here \( W^{k,p}(\Omega) \) denotes, as usual, the Sobolev space of

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functions whose derivatives up to order \( k \) are also \( L^p(\Omega) \) functions.

We say that \((1.1),(1.2)\) is a regular system \((R)\) if a weak solution \( u \) belongs to the space \([C^{1,\alpha}(\Omega)]^m\), where, of course, \( C^{1,\alpha}(\Omega) \) is the space of continuously differentiable functions in \( \Omega \) whose derivatives are locally \( \alpha \)-Hölder continuous.

The history of the regularity problem is described in the book by O.A. Ladyženskaja, N.N. Ural'ceva \([1]\), in the paper by Ch.B. Morrey \([2]\) and elsewhere. It is well known from the result of E. De Giorgi \([3]\) that, for \( m = 1 \), the single equation \((1.1),(1.2)\) is regular. By virtue of a counterexample of J. Nečas \([4]\), there exist systems \((1.1),(1.2)\) which are not regular for \( n \geq 5 \); this question is still open for \( n = 3,4 \). Sufficient conditions for the regularity are also of interest, see M. Giaquinta \([5]\), J. Nečas \([6]\). Since the examples of the regularity are \([W^{1,\infty}(R^n)]^m\) solutions to a system

\[
(1.3) \quad - \frac{\partial}{\partial x_i} \left[a_i^j(\nabla u)\right] = 0
\]

of the type

\[
(1.4) \quad |x - x_0| \leq \varepsilon \left(\frac{x_0 - x}{|x - x_0|}\right),
\]

and in virtue of a trivial fact that a \( C^1(R^n) \) vector function of the type \((1.4)\) is a polynomial of at most first degree, we see that the regularity implies weak Liouville's property: we say that the system \((1.1),(1.2)\) has weak Liouville's property \((WL)\), if for every \( x^0 \in \Omega \), \( \xi \in R^m \), every function \( v \) with a bounded gradient of the type \((1.4)\), solving in \( R^n \) the system

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is a polynomial (more exactly, a vector of polynomials) of at
most first degree. We speak about Liouville's property (L) if
the same is true without supposing (1.4).

We prove in this paper by the "partial regularity" meth-
method, see Ch. B. Morrey [7], E. Giusti, M. Miranda [8], E. Giu-
sti [9], M. Giaquinta [5], that (L) \( \implies \) (R). In this connection
31 relations can be thought of between (R), (WL), (L) (some
are trivial), especially (WL) \( \equiv \) (R), (R) \( \equiv \) (L).

Considering the solutions to (1.3) in the form (1.4), we
can get, see J. Nečas [6], that, for \( m = 1, n \geq 2 \), we have (WL).
Because there is still some hope that for \( n = 3,4 \) we get (R)
for the systems (1.1),(1.2) it is not unthinkable that we ha-
we the property (L) for \( n = 3,4 \), which would be a way to
prove this conjecture.

Clearly there are many other interesting questions, as,
for example, how to avoid the condition \( u \in [W^{1,\infty}(\Omega)]^m \); this
seems to be possible via some growth conditions.

We also prove (in § 3 of this paper) an easy result that
for the systems (1.1),(1.2) and for \( n = 2 \), the property (L)
is satisfied. So we get once more the known result that for
\( n = 2 \) we have (R).

\[ \frac{\partial}{\partial x_i} [a_i^r(x, \xi, \nabla u)] = 0, \]

§ 2. Lemmas. Let us first introduce some notation:

\[ u_{x_0, R} = \frac{1}{\operatorname{mes} B_R(x_0)} \int_{B_R(x_0)} u(x)dx, \]

where \( B_R(x_0) \) is the ball with the center \( x_0 \) and the radius \( R \)
Let us mention the result of S. Campanato [10]: if

$$u \in [W^{1,2}_1(B(0,1)) \cap L^2(B(0,1))]^m$$

is a weak solution to the equation with constant coefficients

$$\int_{B(0,1)} b_{ij}^{hk} D_i u_h^i D_j \psi_k \, dx = 0, \quad \forall \psi \in \mathcal{D}(B(0,1))^m,$$

then for every $0 < \varrho < 1$ we have

$$(*) \quad U(0, \varrho) \leq c \varrho^2 U(0,1),$$

where $c$ depends on $\max |b_{ij}^{hk}|$

and on the constant $\alpha$ of ellipticity:

$$b_{ij}^{hk} \eta_i^h \eta_j^k \geq \alpha |\eta|^2.$$

First we get a modification of the main lemma from [8], [9],[5].

**Lemma 2.1.** Let $v \in \left[ L^\infty(\Omega) \right]^N \cap \left[ W^{1,2}(\Omega) \right]^N$ be a weak solution to the system

$$\int_{\Omega} \left[ A_{ij}^{hk}(x,v) D_i v_h D_j \varphi_k + A_i^{hk}(x,v) D_i v_h \varphi_k \right] \, dx = 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $A_{ij}^{hk}(x,\varphi)$, $A_i^{hk}(x,\varphi)$ are continuous functions in $\bar{\Omega} \times \mathbb{R}^n$, $\varphi_j \in L_p(\Omega)$, $\varphi^k \in L_{\frac{p}{2}}(\Omega)$, $p > n$, $\varphi \neq 0$.

If $x_0 \in \Omega$ and $R \geq \text{dist}(x_0, \partial \Omega)$ we put $v = v^* + w$, where $w \in H^1_0(B(x_0,R))$ is a solution to

$$\int_{B(x_0,R)} \left[ A_{ij}^{hk}(x,v) D_i w_h D_j \varphi_k + A_i^{hk}(x,v) D_i w_h \varphi_k \right] \, dx =$$

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Then for every \( \tau, 0 < \tau < 1 \), there exist \( \varepsilon_0 = \varepsilon_0(\tau, |\nu|_\infty) \), \( R_0 = R_0(\tau, |\nu|_\infty) \) such that if \( R \leq \min(R_0, \text{dist}(x_0, \partial \Omega)) \) and if

\[
V^*(x_0, R) < \varepsilon_0^2
\]

then

\[
V^*(x_0, \tau R) \leq 2c \tau^2 V^*(x_0, R),
\]

where the constant \( c \) is from \((*)\).

**Proof.** Let us suppose the contrary. Then \( \exists \tau, x_\nu \in \Omega, \varepsilon_\nu \rightarrow 0, R_\nu \rightarrow 0, v_\nu^* \in [H^1(\Omega)]^N, |v_\nu^*|_{L^\infty} \leq |\nu|_{L^\infty}, \) such that \( v_\nu^*(x_\nu, R_\nu) = \varepsilon_\nu^2, v_\nu^*(x_\nu, ?R_\nu) > 2c \tau^2 \varepsilon_\nu^2 \)

Put \( x = x_\nu + R_\nu y, s^\nu (y) = \varepsilon_\nu^{-1} [v_\nu^*(x_\nu + R_\nu y) - v_\nu^*(x_\nu, R_\nu)] \).

We have \( \int_{B(0,1)} |s^\nu(y)|^2 dy = s^\nu(0,1) = 1, \) \( (2.5) \)

\[
S^\nu(0, \tau) > 2c \tau^2.
\]

Put further \( t^\nu(y) = \omega^\nu(x_\nu + R_\nu y) \) \((v_\nu^* = v_\nu^* + \omega^\nu)\). Then we can suppose \( x_\nu \rightarrow x_0 \in \Omega, s^\nu \rightarrow s \) in \( L^2(B(0,1)), \)

\( \varepsilon_\nu^2 s^\nu(y) \rightarrow 0 \) almost everywhere in \( B(0,1) \). We have

\[
(2.6) \quad V^*(x_\nu + R_\nu y) = s^\nu (y) \varepsilon_\nu + v_\nu^*(x_\nu, R_\nu) + t^\nu(y).
\]

Since

\[
(2.7) \quad \int_{B(x_\nu, R_\nu)} |\omega^\nu|^2 dx \leq c_1 R_\nu^2 \int_{B(x_\nu, R_\nu)} D_1 \omega h D_1 \omega h dx,
\]

we first get that \((2.2)\) is uniquely solvable for \( R_\nu \) small enough. We further get from \((2.7)\) and \((2.2)\) that

\[
(2.8) \quad \int_{B(0,1)} |t^\nu(y)|^2 dy \leq c_2 R^2(1 - \frac{1}{k^2}),
\]

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so we can also suppose that \( t_\rightarrow (y) \rightarrow 0 \) almost everywhere.
Hence from (2.6) it follows that we can suppose \( \nu_{x,\nu, R} \rightarrow \nu_\rightarrow \in \mathbb{R}^N \) and therefore

\[
A_{ij}(x, y, \nu_\rightarrow, \nu, R, \nu) + t_\rightarrow(y) \rightarrow A_{ij}(x_\rightarrow, \nu_\rightarrow)
\]

almost everywhere in \( B(0, 1) \). Hence we get that \( s_\rightarrow \rightarrow s \) in \( \mathcal{M}_{1, 2}(B(0, 1)) \) and that

\[
(2.9) \int_{B(0, 1)} A_{ij}(x_\rightarrow, \nu_\rightarrow) D_i u D_j \psi_k \, dy = 0 \quad \forall \psi \in \mathcal{D}(B(0, 1)) \].

Thus we have

\[
(2.10) \quad S(0, \tau) \leq c \tau^2 \quad S(0, 1) \leq c \tau^2,
\]

which is a contradiction with

\[
(2.11) \quad S(0, \tau) > 2c \tau^2
\]

obtained from (2.5).

**Lemma 2.2.** Under the conditions of Lemma 2.1, for every point \( x_\rightarrow \in \Omega \) such that \( \nu_\rightarrow(x_\rightarrow, R) < \tau^2 \), there exists a \( B(x_\rightarrow, \tau_1) \subset \Omega \) such that \( \nu \in C^\infty(B(x_\rightarrow, \tau_1)) \) with \( \alpha = \min(\frac{1}{2}, 1 - \frac{R}{p}) \).

**Proof.** We get by a standard argument that if \( \sigma^\rightarrow > 0 \) is small enough, \( \tau - x_\rightarrow < \sigma^\rightarrow \), and \( R_\rightarrow = R - \tau - \tau - x_\rightarrow \), then \( \nu_\rightarrow((\tau_\rightarrow, R_\rightarrow)) < \tau^2 \). If \( \nu = \nu_\rightarrow + \omega \) in \( B(\tau_\rightarrow, R_\rightarrow) \), we first have

\[
(2.12) \int_{B(\tau_\rightarrow, R_\rightarrow)} |\omega|^2 \, dx + c_1 R_\rightarrow^2 \int_{B(\tau_\rightarrow, R_\rightarrow)} |D\omega|^2 \, dx \leq
\]

\[
\leq c_2 R_\rightarrow^2 \int_{B(\tau_\rightarrow, R_\rightarrow)} R^2 \omega^2 \left( \sum \left( \int_{B} | \sigma^\rightarrow |^p \, dx \right)^{\frac{1}{p-1}} \right) \frac{1}{p-1} +
\]

\[
+ \sum \left( \int_{B} | \sigma^\rightarrow |^p \, dx \right)^{\frac{1}{p-1}} \leq c_3 R_\rightarrow^{2m - \frac{2m}{p-1}}.
\]

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Thus

\[(2.13) \quad V(\mathbf{x}, \tau_R^\mathbf{x}) \leq 2V^*(\mathbf{x}, \tau_R^\mathbf{x}) + 2\Omega(\mathbf{x}, \tau_R^\mathbf{x}) \leq
\]
\[\leq 4c \tau^2 V^*(\mathbf{x}, R^\mathbf{x}) + \tau^{-1} R^\mathbf{x} \tau^{-n} \leq
\]
\[\leq 8c \tau^2 V(\mathbf{x}, R^\mathbf{x}) + 8c \tau^2 c_3 R^2 \tau^{-n} + 2c_3 \tau^{-n} R^\mathbf{x} 2^{1-\frac{2}{p}}.
\]

Choose \(\tau \in (0, 1)\) such that \(8c \tau = \tau \leq 1\) and small enough.

We get from (2.13) that

\[(2.14) \quad V(\mathbf{x}, \tau_R^\mathbf{x}) \leq 8c \tau^2 V(\mathbf{x}, R^\mathbf{x}) + c_4 R^\mathbf{x} 2^{1-\frac{2}{p}}.
\]

For \(k\) being a positive integer, we get from (2.14) that

\[(2.15) \quad V(\mathbf{x}, \tau^{k}R^\mathbf{x}) \leq \tau^k V(\mathbf{x}, R^\mathbf{x}) +
\]
\[+ R^\mathbf{x} 2^{1-\frac{2}{p}} \left( \frac{\tau^k}{4} \right) + \frac{c_4 R^\mathbf{x} 2^{1-\frac{2}{p}}}{|\tau^k - \tau^{2k}(1-\frac{2}{p})|} .
\]

If \(0 < \varphi < R - \varphi\) and if we choose \(k\) such that \(\tau^{k+1}R^\mathbf{x} < \varphi \leq \tau^{k}R^\mathbf{x}\), we get \(\tau^k V(\mathbf{x}, \varphi) \leq \left( \frac{\varphi}{\tau^{k}R^\mathbf{x}} \right)^m V(\mathbf{x}, \varphi) \leq V(\mathbf{x}, \tau^{k}R^\mathbf{x}) \leq
\]
\[\leq \frac{\varphi}{\tau^{k}R^\mathbf{x}^2} V(\mathbf{x}, R^\mathbf{x}) + c_4 R^\mathbf{x} 2^{1-\frac{2}{p}} \left( \frac{\varphi}{R^\mathbf{x}^2} \right) + \left( \frac{\varphi}{R^\mathbf{x}^2} \right) 2^{1-\frac{2}{p}} , \text{ and using}
\]
\[|\tau - \tau^2(1-\frac{2}{p})|.
\]

[10], we get the result, q.e.d.

§ 3. Main results

Theorem 1. Let \(u \in [W^1, \infty(\Omega)]^m\) be a weak solution to (1.1) and let the conditions on \(a_1^\mathbf{r}, a_1^\mathbf{r}, a_1^\mathbf{r}, a_1^\mathbf{r}, \Omega\), mentioned in § 1, be fulfilled. Let the system (1.1) satisfy the Liouville's property, i.e., for \(\forall \mathbf{x} \in \Omega\) and \(\forall \varphi \in \mathbb{R}^m\) the only solution to (1.5) defined in the whole \(\mathbb{R}^n\) and possessing a bounded gradient is a polynomial of at most first degree.
Then $u \in C^1(\Omega \setminus K)$, $\alpha = \min\left(\frac{1}{2}, 1 - \frac{R}{p}\right)$.

**Proof.** Let $x^0 \in \Omega$. Put $u_R(y) = \frac{1}{R}[u(x^0 + Ry) - u(x^0)]$, $x^0 + Ry = x$. If $0$ is the image of $\Omega$ we have

\[
\begin{align*}
(3.1) & \quad \int \left[a_1^R(x^0 + Ry, Ru_R(y) + u(x^0), \nabla y u_R(y)) \frac{\partial \psi}{\partial y_i} + \\
& \quad + a^R(x^0 + Ry, Ru_R(y) + u(x^0), \nabla y u_R(y)) R \psi_T(y) \right] dy = \\
& = \int \left[f_1^R(x^0 + Ry) \frac{\partial \psi}{\partial y_i} + f^R(x^0 + Ry) R \psi_T(y) \right] dy.
\end{align*}
\]

Let $B(0,a) \subset \mathbb{R}$. We get in a standard way that

\[
(3.2) \quad \int_{B(0,a)} |D^2 u_R(y)|^2 dy \leq c(a).
\]

Hence we can choose $R_k \to 0$ in such a way, that $u_{R_k} \to p$ in $[W^{1,2}(B(0,a))]^m \forall a > 0$. Thus $p \in [W^{1,\infty}(\mathbb{R}^n)]^m$ and it is a weak solution to

\[
(3.3) \quad \int_{R^n} a_1^R(x^0, u(x^0)), \nabla y p) \frac{\partial \psi}{\partial y_i} dy = 0
\]

$\forall \psi \in [\mathcal{D}(\mathbb{R}^n)]^m$.

Therefore, by assumption, $p$ is a polynomial of at most first degree. So we have

\[
(3.3') \quad 0 \leftarrow \int_{B(0,1)} |Du_{R_k}(y) - Dp|^2 dy = R_k^{-n} \int_{B(x^0, R_k)} |Du(x) - Dp|^2 dx.
\]

If $'$ is the $\frac{\partial}{\partial x_t}$ derivative we get from (1.1) the equation in variations

\[
(3.4) \quad \frac{\partial}{\partial t} \left[ \frac{\partial u^R}{\partial x_j} \frac{\partial \psi}{\partial y_i} + \frac{\partial a^R_{ijx}}{\partial x_j} \frac{\partial \psi}{\partial y_i} + \frac{\partial a^R_{ijy}}{\partial x_j} \frac{\partial \psi}{\partial y_i} + \frac{\partial a^R_{ijy_t}}{\partial x_j} \frac{\partial \psi}{\partial y_i} + \right]
\]
Writing (3.4) for every \( \frac{\partial}{\partial x_t} \), \( t = 1, 2, \ldots, n \), removing the
terms \( \frac{\partial a^k}{\partial u^k}, \frac{\partial \varphi_x}{\partial u^k}, \frac{\partial a^k}{\partial \psi_x}, \frac{\partial a^k}{\partial \varphi_x}, \frac{\partial a^k}{\partial \psi_x} \)
to the right-hand side of (3.4), and denoting by \( v \| \) the de-
rivatives \( \frac{\partial a^k}{\partial u^k} \), we get, with \( \frac{\partial a^k}{\partial u^k} \) (and the same with \( a^k \)), a system of the type (2.1). The re-
sult follows from Lemmas 2.1, 2.2 and from (3.3'), because,
in decomposing \( v = v^* + \omega \) on \( B(x^0, R) \) as in Lemma 2.1, we ha-
ve \( \Omega (x^0, R) \to 0 \) for \( R \to 0 \), as above, so \( v^* (x^0, R) \to 0 \),
q.e.d.

**Theorem 2.** Let us consider the system (1.1), (1.2). Let
\( n \) be the dimension of the space, \( n = 2 \). Then (L) is satis-
fi ed.

**Proof.** Let \( v \in [W^1, \infty (\mathbb{R}^2)]^m \) be a weak solution to the equation

\[
(3.5) \quad \int_{\mathbb{R}^2} a^1_i (x^0, \xi, \nabla v) \frac{\partial \psi_i}{\partial \psi_i} dy = 0.
\]

Let \( T > 0 \) and let \( \eta \in \mathcal{D} (B(0, 2T)), 0 \leq \eta \leq 1, \eta = 1 \) in
\( B(0, T), |D_1 \eta| \leq \frac{\eta}{T} \). We get the equation in variations

\[
(3.6) \quad \int_{\mathbb{R}^2} \frac{\partial a^k}{\partial u^k} (x^0, \xi, \nabla v) \frac{\partial \psi'_i}{\partial \psi'_i} \frac{\partial \psi}{\partial \psi'_i} dy = 0.
\]

Putting \( \psi^*_r = v^*_r \eta^2 \), we get from (3.6), using the bounded-
ness of the gradient, that
(3.7) \[ \int_{B(0, 2T)} |Dv'|^2 dy \leq c_2. \]

Hence \[ \int_{\mathbb{R}^2} |Dv'|^2 dy < \infty. \] But there exists \( \psi^n \in [\mathcal{D}(\mathbb{R}^2)]^m \)
such that \( D\psi^n \rightarrow Dv' \) in \([L^2(\mathbb{R}^2)]^m \) (and there exists \( \Lambda^n \in \mathbb{R}^m \)
such that \( \Lambda^n + \psi^n \rightarrow v' \) in \([L^2_{\text{loc}}(\mathbb{R}^2)]^m \). Hence

\[
\int_{\mathbb{R}^1} \frac{\partial \alpha_{ij}}{\partial v^n}(x^i, \xi, \nabla v) \frac{\partial v'_{\xi_j}}{\partial \psi^n} \frac{\partial v'_{\xi_k}}{\partial \psi^n} dy = 0
\]

and thus \( v \) is a polynomial of at most first degree.

References


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