Jan Malý
A note on separation of sets by approximately continuous functions

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 20 (1979), No. 2, 207--212

Persistent URL: [http://dml.cz/dmlcz/105922](http://dml.cz/dmlcz/105922)

**Terms of use:**

© Charles University in Prague, Faculty of Mathematics and Physics, 1979

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
A NOTE ON SEPARATION OF SETS BY APPROXIMATELY CONTINUOUS FUNCTIONS
Jan MALY

Abstract: An example of two $G_\sigma$-sets with disjoint closures in density topology, which cannot be separated by any approximately continuous function is given.

Key words: Density topology, d-derivative of a set, separation properties of approximately continuous functions.

AMS: Primary 26A15
Secondary 54D15

According to Z. Zahorski [4], given any $G_\sigma$-d-closed (i.e. closed in the density topology) set $A \subset \mathbb{R}$ there exists a bounded approximately continuous function $f$ such that $A = \{ x : f(x) = 0 \}$. Consequently, for every pair $A$, $B$ of disjoint $G_\sigma$-d-closed sets there is an approximately continuous function $f$, which separates $A$ and $B$ in the sense that $0 \leq f \leq 1$, $f = 0$ on $A$, $f = 1$ on $B$.

The last assertion is not generally true, if we suppose $A$, $B$ to be $G_\sigma$-sets with disjoint d-closures only, as follows from the example, given in this paper. This answers negatively to the problem posed by M. Laczkovich [2].

Denote by $\lambda$ the Lebesgue measure and by $\lambda^*$ the corresponding outer measure. If $E \subset \mathbb{R}$ is an arbitrary set
and $x \in \mathbb{R}$, then we define the outer density of $A$ at $x$ by
\[
D(E,x) = \lim_{2h \to 0} \frac{\lambda^n((x-h, x+h) \cap E)}{2h}
\]
and the inner density by $d(E,x) = 1 - D(R - E, x)$. The collection of all sets having the inner density one in each its point forms topology, which will be called the density topology (d-topology). It is easy to see that the d-derivative of a set $E \subset \mathbb{R}$ will be the set $D E = \{ x \in \mathbb{R} : D(E, x) > 0 \}$ and the d-closure of $E$ will be $E \cup D E$.

**Lemma 1.** For an arbitrary bounded interval $I = (a,b)$ and $c \in (0,1)$ there is an open set $G(I,c) \subset I$ with the following properties:

1. $\{a,b\} \in D G(I,c)$.
2. If $x \in \mathbb{R} - I$ and $h > 0$, then $\lambda(G(I,c) \cap (x-h, x+h)) \leq 2ch$.

**Proof.** Put
\[
d_n = c((n+1)^{-1} - (n+2)^{-1}),
\]
\[
L = \bigcup_{n=1}^{\infty} (a+n^{-1} - d_n, a + n^{-1}) \cap (a, a + \frac{1}{2} c(b-a)),
\]
\[
R = \bigcup_{n=1}^{\infty} (b-n^{-1}, b - n^{-1} + d_n) \cap (b - \frac{1}{2} c(b-a), b),
\]
\[
G(I,c) = L \cup R.
\]
The property (1) is evidently satisfied, concretely
\[
D(G(I,c), a) = D(G(I,c), b) = \frac{1}{2} c
\]
(choose $h = n^{-1}$, $n = 1, 2, \ldots$). We shall prove (2) for $x \neq a$.

We claim

3. $\lambda((x-h, x+h) \cap L) \leq ch$. 

- 208 -
Indeed, consider \( m \in \mathbb{N}, (m+1)^{-1} \leq h \leq m^{-1} \). Then
\[
\lambda((x-h, x+h) \cap L) \leq \lambda((a-h, a+h) \cap L) \leq \lambda((a-m^{-1}, a+m^{-1}) \cap L) = c(m+1)^{-1} \leq \varepsilon.
\]

On the other hand,
\[
\lambda((x-h, x+h) \cap R) \leq \varepsilon.
\]

If \( h < \frac{1}{2}(b-a) \), then (4) holds trivially, since
\[
(x-h, x+h) \cap R = \emptyset.
\]

If \( h \geq \frac{1}{2}(b-a) \), then
\[
\lambda((x-h, x+h) \cap R) \leq \lambda R \leq \frac{1}{2} c(b-a) \leq \varepsilon.
\]

From (3) and (4) we immediately obtain (2).

Denote by \( C \) the Cantor's discontinuum (or an arbitrary perfect nowhere dense set with \( \lambda C = 0, \inf C = 0, \sup C = 1 \)). There are open disjoint intervals \((a_i, b_i)\) \((i = 1, 2, ...)\) such that
\[
C = (0,1) - \bigcup_{i=1}^{\infty} (a_i, b_i).
\]

The set \( \bigcup_{i=1}^{\infty} (a_i, b_i) \) will be denoted by \( S \). Further put \( B = C - S \). Finally, consider
\[
A = \bigcup_{i=1}^{\infty} G((a_i, b_i), 2^{-i}).
\]

The sets \( A, B \) and \( S \) have the following important properties:

**Lemma 2.** (i) \( \mathcal{D} A \cap B = \emptyset \).

(ii) \( S \subseteq \mathcal{D} A \).

(iii) \( S \) is not a \( G_\delta \).

(iv) \( A \) and \( B \) are \( G_\delta \) sets with disjoint d-closures.

**Proof.** (i) Let \( x \in B \). Choose \( \varepsilon > 0 \). Find a positive integer \( k \) with \( 2^{-k} < \varepsilon \). There is \( \delta > 0 \) such that
\[ \langle x - \delta', x + \delta' \rangle \cap \bigcup_{i=1}^{k} (a_i, b_i) = \emptyset. \]

For every \( i > k+1 \) and \( h \), \( 0 < h < \sigma' \) we have

\[ \lambda (\langle x-h, x+h \rangle \cap G((a_i, b_i), 2^{-i}) \leq 2^{-i+1}h. \]

Thus

\[ \lambda (\langle x-h, x+h \rangle \cap A) \leq \frac{\delta}{i \cdot 2^{k+1}} 2^{-i+1}h \leq 2 \varepsilon h. \]

Since \( \varepsilon > 0 \) may be chosen arbitrary, \( x \notin \mathcal{D} A \).

(ii) For every \( i = 1,2, \ldots \) we obtain from (i)

\[ \{a_i, b_i\} \subset \mathcal{D} G((a_i, b_i), 2^{-i}) \subset \mathcal{D} A. \]

(iii) The set \( S \) is of the first category and dense in the Baire space \( C \), and thus it is not a \( G_{\sigma'} \).

(iv) Obviously, \( A \) is open and \( \langle 0, 1 \rangle - B = \bigcup_{i \in \mathbb{N}} \langle a_i, b_i \rangle \).

Since \( \lambda B = 0 \), we have \( \mathcal{D} B = \emptyset \). Clearly, \( A \cap B = \emptyset \) and using

(i) we obtain

\[ \text{cl}_{\mathcal{D}} A \cap \text{cl}_{\mathcal{D}} B = \text{cl}_{\mathcal{D}} A \cap B = \mathcal{D} A \cap B = \emptyset. \]

We shall show that there exists a set whose \( d \)-derivative

is not a \( G_{\sigma'} \):

Theorem 1. If \( A \) is as above, then \( \mathcal{D} A \) is not a \( G_{\sigma'} \).

Proof. It is an easy consequence of Lemma 2, parts (i),

(ii) and (iii).

Definition. A function \( f : \mathbb{R} \to \mathbb{R} \) is said to be approximately continuous if for every \( x \in \mathbb{R} \) there is a set \( M \)
such that \( x \in M \), \( d(x, M) = 1 \) and \( f|_M \) is continuous at \( x \).

The approximately continuous functions are just the continuous mappings from the density topology to the euclidean
one. Thus,

(5) \( \{ x : f(x) = 0 \} \) is d-closed

for any approximately continuous function \( f \).

Since any approximately continuous function \( f \) is of the Baire class one (see for example [3]), it follows that

(6) \( \{ x : f(x) = 0 \} \) is a \( G^r \).

**Theorem 2.** Assume that an approximately continuous function \( f \) vanishes on \( A \). Then there exists \( x \in B \) with \( f(x) = 0 \).

(\( A, B \) are as above.)

**Proof.** Denote \( M = \{ x : f(x) = 0 \} \). By (5), \( \partial A \subseteq M \) and by (6), \( M \) is a \( G^r \). Thus \( \partial A \cap C \neq M \cap C \) according to Theorem 1. Hence there is a point \( x \in M \cap C - \partial A \cap C - S = B \).

**Corollary.** The sets \( A \) and \( B \) cannot be separated by any approximately continuous function.

**Remark.** It is not difficult to prove that the d-derivative of any set is always a \( G^r \). We have seen that the d-derivative need not be a \( G^r \). On the other hand, it need not be a \( F_\sigma^r \) as well. Indeed, let \( M \) be a measurable set such that \( \lambda (I \cap M) > 0 \) and \( \lambda (I - M) > 0 \) for every interval \( I \). Then either \( M \) or \( R - M \) is not a \( F_\sigma^r \). Let us remark only, that if \( M \) is a set whose d-derivative is not a \( F_\sigma^r \), then the upper symmetric derivative of the function \( x \mapsto \lambda (\langle 0, x \rangle \cap M) \) is not of the first class of Baire (although the upper derivative, or even the upper symmetric derivative of arbitrary function is of the second class of Baire, see e.g. [1]).
References


Matematicko-fyzikální fakulta
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 18.12. 1978)