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THE MEASURE EXTENSION THEOREM FOR SUBADDITIVE PROBABILITY MEASURES IN ORTHOMODULAR σ – CONTINUOUS LATTICES

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Abstract: The assertion stated in the title of the article is proved.

Key words: Probability measures, logics, orthomodular lattices.

AMS: 20A60

Although the measure theory on logics (orthomodular lattices or posets) is topical (see [5]), no measure extension theorem is known. D.A. Kappos presented in [2] as an open problem the possibility of such extension.

There are some results in [1],[3],[4], but for modular lattices only. P. Volauf in [7] showed that the proof of the extension theorem in [3] works in orthomodular lattices, and he proved the extension theorem for orthocomplemented lattices and probability measures using Carathéodory measurability. Bu as P. Volauf as the author assume that the given measure is a valuation. As it is known, measures on logics need not be valuations.

In the paper we prove an extension theorem for subadditive probability measures. Of course, every non-negative valuation is subadditive, hence our result is a little better than the previous known ones.

Notations and notions. If H is a lattice, we shall write $x_n \nearrow x$, if $x_n \leqq x_{n+1}$ (n = 1,2,...) and $x = \bigvee_{n=1}^{\infty} x_n$; similarly for $x_n \searrow x$. A σ-complete lattice will be called σ-continuous, if $x_n \nearrow x$, $y_n \nearrow y$ implies $y_n \wedge y_n \nearrow x \wedge y$ and dually.

A lattice H with the least element 0 and the greatest element 1 is called orthocomplemented, if there is a mapping $\perp : a \to a^{\perp}$, $H \to H$ such that the following properties are satisfied: (i) $(a^{\perp})^{\perp} = a$ for every $a \in H$. (ii) If $a \leqq b$ then $b^{\perp} \leqq a^{\perp}$. (iii) $a \vee a^{\perp} = 1$ for every $a \in H$. An orthocomplemented lattice is called to be an orthomodular lattice if the following condition is satisfied: (iv) If $a \leqq b$ then $b = a \vee \vee (b \wedge a^{\perp})$. Two elements a, $b \in H$ are called orthogonal if $a \leqq b^{\perp}$ or equivalently $b \leqq a^{\perp}$. A subset A of an orthocomplemented lattice H is called an orthocomplemented sublattice of H if a, $b \in A$ implies $a \vee b \in A$, $a^{\perp} \in A$.

Let A be an orthocomplemented sublattice of an orthomodular lattice H. A mapping $\mu : A \to \langle 0, \infty \rangle$ is called a measure if the following statements are satisfied:

α) $\mu(0) = 0$

β) If $a_n \in A$ (n = 1,2,...) and a_n are pairwise orthogonal and $\bigvee_{n=1}^{\infty} a_n \in A$, then

$$\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n).$$

A measure $\mu : A \to \langle 0, \infty \rangle$ is called a probability measure if $\mu(1) = 1$. A measure $\mu : A \to \langle 0, \infty \rangle$ is called subadditive if $\mu(a \vee b) \leqq \mu(a) + \mu(b)$ for every a, $b \in A$.

It is not very difficult to prove (by the help of (iv)) that every measure is non-decreasing and upper continuous (i.e. $a_n \nearrow a \Rightarrow \mu(a_n) \nearrow \mu(a)$).

<u>Construction</u>. We start with an orthocomplemented sublattice A of an orthomodular, σ-continuous lattice H and a subadditive probability measure $\mu : A \to \langle 0,1 \rangle$. We want to extend it to the σ-complete orthocomplete lattice S(A) generated by A.

<u>Lemma</u>. Let $a_n, b_n \in A$ $(n = 1,2,\dots)$, $a_n \nearrow a$, $b_n \nearrow b$, $a \leqq b$. Then $\lim_{n\to\infty} \mu(a_n) \leqq \lim_{n\to\infty} \mu(b_n)$.

<u>Proof</u>. Evidently $a_n \wedge b_m \nearrow a_n \wedge b = a_n$ $(m \to \infty)$, hence $\mu(a_n) = \lim_{m\to\infty} \mu(a_n \wedge b_m) \leqq \lim_{m\to\infty} \mu(b_m)$ and therefore $\lim_{n\to\infty} \mu(a_n) \leqq \lim_{m\to\infty} \mu(b_m)$.

Now put $A^+ = \{b \in H;\ \exists\, a_n \in A,\ a_n \nearrow b\}$. The preceding lemma gives a possibility to define a mapping $\mu^+ : A^+ \to \langle 0, \infty \rangle$ by the formula

$$\mu^+(b) = \lim_{n\to\infty} \mu(a_n),\quad a_n \nearrow b.$$

Then we can put

$$\mu^*(x) = \inf\{\mu^+(b);\ b \in A^+,\ b \geqq x\},\quad x \in H$$

and by such a way we obtain a mapping $\mu^* : H \to \langle 0,1 \rangle$. Similarly they can be defined A^-, μ^-, μ_*. The last step of our construction is the set

$$L = \{x \in H;\ \mu_*(x) = \mu^*(x)\ .$$

Later we prove that $L \supset S(A)$ and $\mu^*/S(A)$ is the asked extension.

It is easy to prove that μ^+, μ^- are extensions of μ, μ^+ is upper continuous, non-decreasing and subadditive. Further μ^* is an extension of μ^+, μ^* is non-decreasing, subadditive and $\mu^*(x) \geq \mu_*(x)$ for every $x \in H$.

Main theorem. Let H be a σ-continuous, orthomodular lattice, A its orthocomplemented sublattice, $\mu: A \longrightarrow \langle 0,1 \rangle$ a subadditive probability measure. Let S(A) be the σ-complete orthocomplemented sublattice of H generated by A. Then there is exactly one measure $\bar{\mu}: S(A) \longrightarrow \langle 0,1 \rangle$ that is an extension of μ . The measure $\bar{\mu}$ is a subadditive probability measure.

Proof. Our main result will be proved by a sequence of propositions.

Proposition 1. Let $x \in H$, $y \in L$, $y \leqq x$. Then $\mu^*(x) = \mu^*(y) + \mu^*(x \wedge y^{\perp})$.

Proof. 1. Let first $a \in A$, $b \in A^+$, $a \leqq b$. Then $\mu^+(b) = \mu(a) + \mu^+(b \wedge a^{\perp})$. Namely, $a \leqq a_n \nearrow b$, $a_n \in A$ implies $\mu(a_n) = \mu(a) + \mu(a_n \wedge a^{\perp})$. Since $a_n \nearrow b$, $a_n \wedge a^{\perp} \nearrow b \wedge a^{\perp}$, we obtain $\mu^+(b) = \mu(a) + \mu^+(b \wedge a^{\perp})$.

2. If $b, d \in A^+$, $d \leqq b$, then $\mu^+(b) \geqq \mu^+(d) + \mu^*(b \wedge d^{\perp})$. Indeed, $d_n \nearrow d$, $d_n \in A$ and 1 imply $\mu^+(b) = \mu(d_n) + \mu^+(b \wedge d_n^{\perp}) \geqq \mu(d_n) + \mu^*(b \wedge d^{\perp})$,which gives $\mu^+(b) = \mu^+(d) + \mu^*(b \wedge d^{\perp})$.

3. If $b \in A^+$, $c \in A^-$, $c \leqq b$, then $\mu^+(b) \geqq \mu^-(c) + \mu^+(b \wedge c^{\perp})$. Take $c_n \in A$, $c_n \searrow c$. Since $b \wedge c_n \in A^+$, $b \wedge c_n \leqq b$ we have by 2

$$\mu^+(b) \geqq \mu^+(b \wedge c_n) + \mu^*(b \wedge (b \wedge c_n)^{\perp}) \geqq \mu^+(b \wedge c_n) + \mu^*(b \wedge c_n^{\perp}) = \mu^+(b \wedge c_n) + \mu^+(b \wedge c_n^{\perp}).$$

Taking $n \longrightarrow \infty$ we obtain

$$\mu^+(b) \geqq \lim_{n \to \infty} \mu^+(b \wedge c_n) + \lim_{n \to \infty} \mu^+(b \wedge c_n^{\perp}) \geqq \mu^-(c) + \mu^+(b \wedge c^{\perp}).$$

4. Let $x \in H$, $c \in A^-$, $c \leqq x$. We prove that $\mu^*(x) \geqq \geqq \mu^-(c) + \mu^*(x \wedge c^{\perp})$. Namely, if $b \in A^+$, $b \geqq x$, then

$\mu^+(b) \geqq \mu^-(c) + \mu^+(b \wedge c^{\perp}) \geqq \mu^-(c) + \mu^*(x \wedge c^{\perp})$,

hence $\mu^*(x) \geqq \mu^-(c) + \mu^*(x \wedge c^{\perp})$, too.

5. Finally we prove the assertion stated in Proposition. Let $x \in H$, $y \in L$, $y \leqq x$. Take $c \leqq y$, $c \in A^-$. By 4 we have $\mu^*(x) \geqq \mu^-(c) + \mu^*(x \wedge c^{\perp}) \geqq \mu^-(c) + \mu^*(x \wedge y^{\perp})$, hence $\mu^*(x) - \mu^*(x \wedge y^{\perp}) \geqq \mu^-(c)$. Therefore

$$\mu^*(x) - \mu^*(x \wedge y^{\perp}) \geqq \mu_*(y) = \mu^*(y).$$

The opposite inequality follows from the subadditivity of μ^*.

Proposition 2. If $y \in L$, then $y^{\perp} \in L$.

Proof. Evidently $\mu^+(b) + \mu^-(b^{\perp}) = 1$ for every $b \in \in A^+$. Let $b \geqq y$. Then $b^{\perp} \leqq y^{\perp}$, hence

$$1 = \mu^+(b) + \mu^-(b^{\perp}) \leqq \mu^+(b) + \mu_*(y^{\perp})$$

therefore

$$1 - \mu_*(y^{\perp}) \leqq \mu^*(y).$$

Proposition 1 gives $(x = 1)$ $1 = \mu^*(y) + \mu^*(y^{\perp})$, hence

$$\mu^*(y) + \mu_*(y^{\perp}) \geqq 1 = \mu^*(y) + \mu^*(y^{\perp})$$

which implies $\mu_*(y^{\perp}) \geqq \mu^*(y^{\perp})$.

Proposition 3. If $z_n \in L$ $(n = 1,2,\ldots)$, $z_n \nearrow z$ (or $z_n \searrow z$ resp.), $z \in H$, then $z \in L$ and $\mu^*(z) = \lim_{n \to \infty} \mu^*(z_n)$.

Proof. Let $z_n \nearrow z$. Put $z_0 = 0$. By Proposition 1

$$\mu^*(z_n) - \mu^*(z_{n-1}) = \mu^*(z_n \wedge z_{n-1}^{\perp}),\ n = 1,2,\ldots .$$

To every $\varepsilon > 0$ there is $y_n \in A^+$, $y_n \geqq z_n \wedge z_{n-1}^{\perp}$ such that

$$\mu^*(z_n \wedge z_{n-1}^{\perp}) > \mu^+(y_n) - \frac{\varepsilon}{2^n}\ , \ n = 1,2, \dots .$$

By adding these inequalities we obtain

$$\mu^*(z_n) > \sum_{i=1}^{n} (\mu^+(y_i) - \frac{\varepsilon}{2^i}) \geqq \mu^+(\bigvee_{i=1}^{n} y_i) - \sum_{i=1}^{n} \frac{\varepsilon}{2^i}$$

and therefore

$$\mu^*(z) \geqq \lim_{n\to\infty} \mu^*(z_n) = \lim_{n\to\infty} \mu^+(\bigvee_{i=1}^{n} y_i) - \varepsilon =$$
$$= \mu^+(\bigvee_{n=1}^{\infty} y_n) - \varepsilon \geqq \mu^*(z) - \varepsilon$$

and the equality $\mu^*(z) = \lim_{n\to\infty} \mu^*(z_n)$ is obtained. Further

$$\mu_*(z) \leqq \mu^*(z) = \lim_{n\to\infty} \mu^*(z_n) = \lim_{n\to\infty} \mu_*(z_n) \leqq \mu_*(z),$$

hence $z \in L$. The second part of Proposition (for non-increasing sequences) follows from Proposition 2 and the first part.

Proposition 4. $\bar{\mu} = \mu^*/L$ is an additive mapping, i.e. $x, y \in L$, $x \leqq y^{\perp}$ implies $\mu^*(x \vee y) = \mu^*(x) + \mu^*(y)$.

Proof. First take $c, d \in A^-$, $c \leqq d^{\perp}$. Then by Proposition 1

$$1 - \mu^-(d) = \mu^+(d^{\perp}) = \mu^*(d^{\perp}) = \mu^-(c) + \mu^*(d^{\perp} \wedge c^{\perp}) =$$
$$= \mu^-(c) + \mu^+((d \vee c)^{\perp}) = \mu^-(c) + 1 - \mu^-(c \vee d).$$

Now let $x, y \in H$, $x \leqq y^{\perp}$, $c, d \in A^-$, $c \leqq x$, $d \leqq y$ such that

$$\mu_*(x) - \varepsilon < \mu^-(c), \quad \mu_*(y) - \varepsilon < \mu^-(d).$$

Of course, $c \leqq x \leqq y^{\perp} \leqq d^{\perp}$, hence

$$\mu_*(x \vee y) \leqq \mu^*(x \vee y) \leqq \mu^*(x) + \mu^*(y) = \mu_*(x) + \mu_*(y) <$$
$$< \mu^-(c) + \mu^-(d) + 2\varepsilon = \mu^-(c \vee d) + 2\varepsilon \leqq \mu_*(x \vee y) + 2\varepsilon .$$

Proposition 5. Let S(A) be the σ-complete orthocomplemented lattice generated by A, M(A) be the least set over A closed under monotone sequences. Then $S(A) = M(A)$.

Proof. It can be proved by a standard way. (See e.g. [3], lemma 1.)

Proof of Main theorem. 1. Existence. Evidently $S(A) = M(A) \subset L$. Put $\bar{\mu} = \mu^*/S(A)$. By Propositions 3 and 4 $\bar{\mu}$ is a measure. $\bar{\mu}$ is a subadditive probability measure since μ has these properties.

2. Uniqueness. Let $\nu : S(A) \to R$ be a measure $\nu/A = \mu$. Put $K = \{x \in S(A);\ \bar{\mu}(x) = \nu(x)\}$. Evidently $K \supset A$, K is closed under limits of monotone sequences. Therefore $K \supset M(A) = S(A)$.

References

[1] GYORFFY L. - RIEČAN B.: On the extension of measures in relatively complemented lattices, Mat. časop. 23 (1973), 158-163.

[2] KAPPOS D.A.: Measure theory on orthocomplemented posets and lattices. In: Measure theory, Oberwolfach 1975, Proceedings, Springer, Berlin, Lecture notes in math., vol. 541, 323-343.

[3] RIEČAN B.: On the extension of a measure on lattices, Mat. čas. 19(1969), 44-49.

[4] RIEČAN B.: A note on the extension of measures on lattices, Mat. časop. 20(1970), 239-244.

[5] VARADARAJAN V.S.: Geometry of quantum theory, Van Nostrand, New York, 1968.

[6] VOLAUF P.: The measure extension problem on ortholattices, Acta math. Univ. Comen., to appear.

[7] VOLAUF P.: Some problems of the probability theory in ordered spaces, PhD dissertation. Bratislava, 1977.

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