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Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 2, 309--316

Persistent URL: http://dml.cz/dmlcz/105929

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THE MEASURE EXTENSION THEOREM FOR SUBADDITIVE
PROBABILITY MEASURES IN ORTHOMODULAR \( \sigma \)– CONTINUOUS
LATTICES
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Abstract: The assertion stated in the title of the ar-
ticle is proved.

Key words: Probability measures, logics, orthomodular
lattices.

AMS: 20A60

Although the measure theory on logics (orthomodular lat-
tices or posets) is topical (see [5]), no measure extension
theorem is known. D.A. Kappas presented in [2] as an open
problem the possibility of such extension.

There are some results in [1],[3],[4], but for modular
lattices only. P. Volauf in [7] showed that the proof of the
extension theorem in [3] works in orthomodular lattices, and
he proved the extension theorem for orthocomplemented latti-
ces and probability measures using Carathéodory measurability.
But as P. Volauf as the author assume that the given mea-
sure is a valuation. As it is known, measures on logics need
not be valuations.

In the paper we prove an extension theorem for subaddi-
tive probability measures. Of course, every non-negative va-
luation is subadditive, hence our result is a little better than the previous known ones.

Notations and notions. If $H$ is a lattice, we shall write $x_n \uparrow x$, if $x_n \leq x_{n+1}$ ($n = 1, 2, \ldots$) and $x = \bigvee_{n \geq 1} x_n$; similarly for $x_n \downarrow x$. A $\sigma$-complete lattice will be called $\sigma$-continuous, if $x_n \uparrow x$, $y_n \uparrow y$ implies $y_n \wedge x \uparrow y$ and dually.

A lattice $H$ with the least element 0 and the greatest element 1 is called orthocomplemented, if there is a mapping $\perp : a \rightarrow a^\perp$, $H \rightarrow H$ such that the following properties are satisfied: (i) $(a^\perp)^\perp = a$ for every $a \in H$. (ii) If $a \not\leq b$ then $b^\perp \not\leq a^\perp$. (iii) $a \vee a^\perp = 1$ for every $a \in H$. An orthocomplemented lattice is called to be an orthomodular lattice if the following condition is satisfied: (iv) If $a \not\leq b$ then $b = a \vee (b \wedge a^\perp)$. Two elements $a$, $b \in H$ are called orthogonal if $a \not\leq b^\perp$ or equivalently $b \not\leq a^\perp$. A subset $A$ of an orthocomplemented lattice $H$ is called an orthocomplemented sublattice of $H$ if $a, b \in A$ implies $a \vee b \in A, a^\perp \in A$.

Let $A$ be an orthocomplemented sublattice of an orthomodular lattice $H$. A mapping $\mu : A \rightarrow (0, \infty)$ is called a measure if the following statements are satisfied:

$\alpha)$ $\mu(0) = 0$

$\beta)$ If $a_n \in A$ ($n = 1, 2, \ldots$) and $a_n$ are pairwise orthogonal and $\bigvee_{n \geq 1} a_n \in A$, then

$$\mu \left( \bigvee_{n \geq 1} a_n \right) = \bigvee_{n \geq 1} \mu(a_n).$$

A measure $\mu : A \rightarrow (0, \infty)$ is called a probability measure if $\mu(1) = 1$. A measure $\mu : A \rightarrow (0, \infty)$ is called subadditive if $\mu(a \vee b) \leq \mu(a) + \mu(b)$ for every $a, b \in A$.

It is not very difficult to prove (by the help of (iv))
that every measure is non-decreasing and upper continuous (i.e. \( a_n \wedge a \Rightarrow \mu(a_n) \leq \mu(a) \)).

**Construction.** We start with an orthocomplemented sub-lattice \( A \) of an orthomodular, 6-continuous lattice \( \mathcal{H} \) and a subadditive probability measure \( \mu: A \rightarrow \langle 0,1 \rangle \). We want to extend it to the 6-complete orthocomplete lattice \( S(A) \) generated by \( A \).

**Lemma.** Let \( a_n, b_n \in A \) (\( n = 1,2,..., \)), \( a_n \wedge b \Rightarrow a, b \Rightarrow b, a \leq b \). Then \( \lim_{n \to \infty} \mu(a_n) \leq \lim_{n \to \infty} \mu(b_n) \).

**Proof.** Evidently \( a_n \wedge b_m \wedge a_n \wedge b = a_n \) (\( m \to \infty \)), hence \( \mu(a_n) = \lim_{m \to \infty} \mu(a_n \wedge b_m) \leq \lim_{m \to \infty} \mu(b_m) \) and therefore \( \lim_{n \to \infty} \mu(a_n) \leq \lim_{m \to \infty} \mu(b_m) \).

Now put \( A^+ = \{ b \in \mathcal{H}; \exists a_n \in A, a_n \wedge b \} \). The preceding lemma gives a possibility to define a mapping \( \mu^+: A^+ \rightarrow \langle 0,\infty \rangle \) by the formula

\[
\mu^+(b) = \lim_{n \to \infty} \mu(a_n), \quad a_n \wedge b.
\]

Then we can put

\[
\mu^*(x) = \inf \{ \mu^+(b); b \in A^+, b \preceq x \}, \quad x \in \mathcal{H}
\]

and by such a way we obtain a mapping \( \mu^*: \mathcal{H} \rightarrow \langle 0,1 \rangle \). Similarly they can be defined \( A^-, \mu^-, \mu^- \). The last step of our construction is the set

\[
L = \{ x \in \mathcal{H}; \mu^*(x) = \mu^*(x) \}
\]

Later we prove that \( L \supseteq S(A) \) and \( \mu^*/S(A) \) is the asked extension.

It is easy to prove that \( \mu^+, \mu^- \) are extensions of \( \mu \), \( \mu^+ \) is upper continuous, non-decreasing and subadditive.

Further \( \mu^* \) is an extension of \( \mu^+, \mu^* \) is non-decreasing,
subadditive and \( \mu^*(x) \geq \mu^-(x) \) for every \( x \in H \).

**Main theorem.** Let \( H \) be a \( \sigma \)-continuous, orthomodular lattice, \( A \) its orthocomplemented sublattice, \( \mu : A \rightarrow (0,1) \) a subadditive probability measure. Let \( S(A) \) be the \( \sigma \)-complete orthocomplemented sublattice of \( H \) generated by \( A \). Then there is exactly one measure \( \tilde{\mu} : S(A) \rightarrow (0,1) \) that is an extension of \( \mu^* \). The measure \( \tilde{\mu} \) is a subadditive probability measure.

**Proof.** Our main result will be proved by a sequence of propositions.

**Proposition 1.** Let \( x \in H \), \( y \in L \), \( y \neq x \). Then \( \mu^*(x) = \mu^*(y) + \mu^*(x \wedge y) \).

**Proof.** 1. Let first \( a \in A \), \( b \in A^+ \), \( a \notin b \). Then \( \mu^+(b) = \mu(a) + \mu(b \wedge a^\perp) \). Namely, \( a \nvdash n \vdash b \), \( a \notin A \) implies 
   \( \mu(a_n) = \mu(a) + \mu(a_n \wedge a^\perp) \). Since \( a_n \vdash b \), \( a_n \wedge a^\perp b \wedge a^\perp \), we obtain 
   \( \mu^+(b) = \mu(a) + \mu(b \wedge a^\perp) \).

2. If \( b, d \in A^+ \), \( d \notin b \), then \( \mu^+(b) \geq \mu^+(d) + \mu^*(b \wedge d^\perp) \).

Indeed, \( d_n \vdash d, d_n \in A \) and \( 1 \) imply 
\( \mu^+(b) = \mu(d_n^+) + \mu^+(b \wedge d_n^\perp) \geq \mu(d_n^+) + \mu^*(b \wedge d^\perp) \), which gives 
\( \mu^+(b) = \mu^+(d) + \mu^*(b \wedge d^\perp) \).

3. If \( b \in A^+ \), \( c \in A^- \), \( c \notin b \), then \( \mu^+(b) \geq \mu^-(c) + \mu^*(b \wedge c^\perp) \). Take \( c_n \in A \), \( c_n \nvdash c \). Since \( b \wedge c_n \in A^+ \), \( b \wedge c_n \notin b \), we have by 2 
\( \mu^+(b) \geq \mu^+(b \wedge c_n) + \mu^*(b \wedge (b \wedge c_n)^\perp) \geq \mu^+(b \wedge c_n) + \mu^*(b \wedge c_n^\perp) \).

Taking \( n \rightarrow \infty \) we obtain
\( \mu^+(b) \geq \lim_{n \rightarrow \infty} \mu^+(b \wedge c_n) + \lim_{n \rightarrow \infty} \mu^+(b \wedge c_n^\perp) \geq \mu^-(c) + \mu^+(b \wedge c^\perp) \).
4. Let $x \in H$, $c \in A^+$, $c \not\in x$. We prove that $\mu^*(x) \geq \mu^-(c) + \mu^*(x \land c \perp)$. Namely, if $b \in A^+$, $b \geq x$, then

$$
\mu^+(b) \geq \mu^-(c) + \mu^+(b \land c \perp) \geq \mu^-(c) + \mu^*(x \land c \perp),
$$

hence $\mu^*(x) \geq \mu^-(c) + \mu^*(x \land c \perp)$, too.

5. Finally we prove the assertion stated in Proposition.

Let $x \in H$, $y \in L$, $y \not\in x$. Take $c = y$, $c \in A^-$. By 4 we have

$$
\mu^*(x) \geq \mu^-(c) + \mu^*(x \land c \perp) \geq \mu^-(c) + \mu^*(x \land y \perp),
$$

hence $\mu^*(x) - \mu^*(x \land y \perp) \geq \mu^-(c)$. Therefore

$$
\mu^*(x) - (\mu^*(x \land y \perp) \geq (\mu^*(y) = \mu^*(y).
$$

The opposite inequality follows from the subadditivity of $\mu^*$.

**Proposition 2.** If $y \in L$, then $y \perp \in L$.

**Proof.** Evidently $\mu^+(b) + \mu^-(b \perp) = 1$ for every $b \in A^+$. Let $b \geq y$. Then $b \perp \leq y \perp$, hence

$$
1 = \mu^+(b) + \mu^-(b \perp) \leq \mu^+(b) + \mu^*(y \perp)
$$

therefore

$$
1 - \mu^*(y \perp) \leq \mu^*(y).
$$

Proposition 1 gives $(x = 1) 1 = \mu^*(y) + \mu^*(y \perp)$, hence

$$
\mu^*(y) + \mu^*(y \perp) \geq 1 = \mu^*(y) + \mu^*(y \perp)
$$

which implies $\mu^*(y \perp) \geq \mu^*(y \perp)$.

**Proposition 3.** If $z_n \in L$ $(n = 1, 2, \ldots)$, $z_n \uparrow z$ (or $z_n \downarrow z$ resp.), $z \in H$, then $z \in L$ and $\mu^*(z) = \lim_{n \to \infty} \mu^*(z_n)$.

**Proof.** Let $z_n \uparrow z$. Put $z_0 = 0$. By Proposition 1

$$
\mu^*(z_n) - \mu^*(z_{n-1}) = \mu^*(z_n \land z_{n-1} \perp), n = 1, 2, \ldots.
$$

To every $\varepsilon > 0$ there is $y_n \in A^+$, $y_n \geq z_n \land z_{n-1} \perp$ such that
By adding these inequalities we obtain

\[ \mu^*(z_n) \geq \frac{\varepsilon}{n} (\mu^+(y_i) - \frac{\varepsilon}{2^n}) \geq \mu^+(\frac{\varepsilon}{i+1} y_i) = \frac{\varepsilon}{2^n} \]

and therefore

\[ \mu^*(z) = \lim_{n \to \infty} \mu^*(z_n) = \lim_{n \to \infty} \mu^+(\frac{\varepsilon}{n+1} y_n) - \varepsilon = \mu^+(\varepsilon y_n) - \varepsilon \geq \mu^*(z) - \varepsilon \]

and the equality \( \mu^*(z) = \lim \mu^*(z_n) \) is obtained. Further

\[ \mu^*(z) = \mu^*(z) = \lim_{n \to \infty} \mu^*(z_n) = \lim_{n \to \infty} \mu^*(z_n) \neq \mu^*(z), \]

hence \( z \in L \). The second part of Proposition (for non-increasing sequences) follows from Proposition 2 and the first part.

**Proposition 4.** \( \mu = \mu^*/L \) is an additive mapping, i.e. \( x, y \in L, x \leq y \perp \) implies \( \mu^*(x \vee y) = \mu^*(x) + \mu^*(y) \).

**Proof.** First take \( c, d \in A^-, c \leq d \perp \). Then by Proposition 1

\[ 1 - \mu^-(d) = \mu^+(d \perp) = \mu^*(d \perp) = \mu^-(c) + \mu^*(d \perp \cap c \perp) = \mu^-(c) + \mu^+((d \lor c) \perp) = \mu^-(c) + 1 - \mu^-(c \lor d) \]

Now let \( x, y \in H, x \leq y \perp, c, d \in A^-, c \leq x, d \leq y \) such that

\[ \mu^*(x) - \varepsilon \leq \mu^-(c), \quad \mu^*(y) - \varepsilon \leq \mu^-(d) \]

Of course, \( c \leq x \leq y \perp \leq d \perp \), hence

\[ \mu^*(x \lor y) \leq \mu^*(x \lor y) \leq \mu^*(x) + \mu^*(y) = \mu^*(x) + \mu^*(y) < \mu^-(c) + \mu^-(d) + 2 \varepsilon = \mu^-(c \lor d) + 2 \varepsilon \leq \mu^*(x \lor y) + 2 \varepsilon \]

**Proposition 5.** Let \( S(A) \) be the \( \mathcal{G} \)-complete orthocomplemented lattice generated by \( A \), \( M(A) \) be the least set over \( A \)
closed under monotone sequences. Then $S(A) = M(A)$.

**Proof.** It can be proved by a standard way. (See e.g. [3], lemma 1.)

**Proof of Main theorem.** 1. Existence. Evidently $S(A) = M(A) \cup L$. Put $\tilde{\mu} = \mu \cdot S(A)$. By Propositions 3 and 4, $\tilde{\mu}$ is a measure. $\tilde{\mu}$ is a subadditive probability measure since $\mu$ has these properties.

2. Uniqueness. Let $\nu : S(A) \to R$ be a measure $\nu / A = \mu$. Put $K = \{x \in S(A) ; \tilde{\mu}(x) = \nu(x)\}$. Evidently $K \supset A$, $K$ is closed under limits of monotone sequences. Therefore $K \supset M(A) = S(A)$.

**References**


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(Oblatum 13.2. 1979)