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PERIODIC SOLUTIONS TO THE INHOMOGENEOUS SINE-GORDON EQUATION

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Abstract: For α and h satisfying certain conditions and for every ε sufficiently close to 0 it is shown that there exists a function $u \in C^2$ which fulfils
$u_{tt} - u_{xx} = \varepsilon(h(t,x) + \alpha \sin u)$, $u(t,0) = u(t,\pi) = 0$ and $u(t + 2\pi, x) = u(t,x)$.

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In [1], O. Vejvoda derived sufficient conditions for the existence of 2π-periodic solutions to the problem

(1) $u_{tt}(t,x) - u_{xx}(t,x) = \varepsilon f(t,x,u,\varepsilon)$, $t \in R$, $x \in \langle 0,\pi \rangle$

(2) $u(t,0) = u(t,\pi) = 0$, $t \in R$

and studied the problem with $f(t,x,u,\varepsilon) = h(t,x) + \alpha u + \beta u^3$ in detail.

In this paper the same problem with $f(t,x,u,\varepsilon) = h(t,x) + \alpha \sin u$ is treated. In the sequel the functions u and f are supposed to be extended in x on R by

$$u(t,x) = -u(t,-x) = u(t,x + 2\pi),\ f(t,x,u,\varepsilon) = -f(t,-x,-u,\varepsilon) = f(t,x + 2\pi,u,\varepsilon).$$

The extended functions will be denoted again by u and f. Let us note that if $u(t,x) = u(t, \pi - x)$ then $u(t,x+\pi) = -u(t,x)$. Put

$$C^{2*}_{2\pi}(R) = \{ s \in C^2(R);\ s(x+\pi) = -s(x) \},$$

$$C^{2*}_{e}([0,2\pi] \times R) = \{ u \in C^2([0,2\pi] \times R);\ u(t,x) = -u(t,-x) = u(t, \pi - x) \},$$

$$C^{2*}_{2\pi, e}(R^2) = \{ u \in C^2(R^2);\ u(t,x) = u(t+2\pi, x) = -u(t,-x) = u(t, x+2\pi) = u(t, \pi - x) \}$$

and equip these spaces with the usual norms in which they are Banach spaces.

Let us first recall the result which is the starting point of our investigation.

Theorem 1 (cf. Theorem 4.1.3 in [1]).

(i) Let a function f be continuous together with its derivatives

$$\frac{\partial^{j+k} f}{\partial x^j u^k},\ j+k \le 3,\ j \le 2$$

on $R \times \langle 0, \pi \rangle \times R \times \langle -\varepsilon_0, \varepsilon_0 \rangle$, $\varepsilon_0 > 0$.

(ii) Let $f(t,0,0,\varepsilon) = f(t,\pi,0,\varepsilon) =$

$$\frac{\partial^2 f}{\partial x^j u^k}(t,0,0,\varepsilon) = \frac{\partial^2 f}{\partial x^j u^k}(t,\pi,0,\varepsilon) = 0,$$

$j + k = 2$.

(iii) Let $f(t,x,u,\varepsilon)$ be 2π-periodic in t.

(iv) Let the equation

(3) $$G(s)(x) = \int_0^{2\pi} f(\pi, x-\tau, s(x) - s(2\tau - x), 0)\, d\tau = 0$$

have a solution $s^* \in C^{2*}_{2\pi}(R)$,

(v) Let there exist $[G'_s(s^*)]^{-1} \in L(C^{2*}_{2\pi}(R), C^{2*}_{2\pi}(R))$ where $C^{2*}_{2\pi}(R) \supset \mathbb{R}(\Gamma)$, while

$$\Gamma(u,\varepsilon)(x) = \int_0^{2\pi} f(\tau, x-\tau, u(\tau, x-\tau), \varepsilon)\,d\tau ,$$

$$u \in C^{2*}_e([0,2\pi] \times R),\ \varepsilon \in \langle -\varepsilon_0, \varepsilon_0 \rangle .$$

Then for sufficiently small ε the problem (1),(2) has a solution $u^* \in C^{2*}_{2\pi,e}(R^2)$.

Our aim is to prove the following theorem:

<u>Theorem 2.</u>

(a) Let $h(t,x)$ together with its derivatives $\frac{\partial h}{\partial x}$, $\frac{\partial^2 h}{\partial x^2}$ be continuous on $R \times \langle 0, \pi \rangle$.

(b) Let $h(t,0) = h(t,\pi) = \frac{\partial^2 h}{\partial x^2}(t,0) = \frac{\partial^2 h}{\partial x^2}(t,\pi) = 0$ and $h(t, \pi - x) = h(t,x)$.

(c) Let h be 2π -periodic in t.

(d) If $H(x) = \int_0^{2\pi} h(\tau, x-\tau)\,d\tau \not\equiv 0$, then let

$$\alpha \geq 2 \cdot \|H\|_0^2 \cdot \left[\int_0^{2\pi} \sqrt{2 \cdot \|H\|_0^2 - H^2(x)}\,dx\right]^{-1},$$

where $\|H\|_0 = \max_{x \in R} |H(x)|$.

Then for sufficiently small ε the problem (1),(2) with $f(t,x,u,\varepsilon) = h(t,x) + \alpha \sin u$ has a solution $u \in C^{2*}_{2\pi,e}(R^2)$.

<u>Proof</u>: The assumptions (i),(ii),(iii) of Theorem 1 are immediate consequences of the hypotheses (a),(b),(c) of the present theorem.

(iv) Denoting $I = \int_0^{2\pi} \cos s(\xi) d\xi$, we rearrange the equation (3) into the form

(4) $G(s)(x) = \alpha I \sin s(x) + H(x) = 0.$

For a while let us consider the functional I as a known constant. If $H(x) \equiv 0$, put $s^*(x) \equiv 0$. In the opposite case let us suppose that $|H(x)| < |\alpha I|$ and put $\hat{s}(I,x) = -\arcsin (\alpha I)^{-1} H(x)$. Clearly $\hat{s}$ is a solution to (4) if and only if

(5) $I = \int_0^{2\pi} \cos \hat{s}(I, \xi) d\xi = \int_0^{2\pi} (1-(\alpha I)^{-2} H^2(x))^{1/2} d x \equiv p(I).$

Evidently $p(2\pi) < 2\pi$ and (d) implies $p(\sqrt{2}\alpha^{-1} \| H\|_0) \geq \geq \sqrt{2}\alpha^{-1} \| H\|_0$. So (5) has at least one solution $I = I^*$ satisfying $|I^*| \geq \alpha^{-1} \| H\|_0$. Setting $s^*(x) = \hat{s}(I^*, x) \in \in C^{2*}_{2\pi}(R)$ we obtain a solution to (4) and the assumption (iv) is verified.

To prove (v) it suffices to show that for every $\wp \in C^{2*}_{2\pi}(R)$, the equation

$$G_s'(s^*)(\sigma)(x) \equiv \alpha I^* \sigma(x) \cos s^*(x) - \alpha J \sin s^*(x) = \\ = \wp(x) \in C^{2*}_{2\pi}(R),$$

where $J = \int_0^{2\pi} \sin s^*(\xi) . \sigma(\xi) d\xi$, has a unique solution $\sigma(x) \in C^{2*}_{2\pi}(R)$ with $\|\sigma\|_2 \leq C \|\wp\|_2$, C being a constant. We obtain easily that

$$\sigma(x) = (\wp(x) + \alpha J^* \sin s^*(x)) (\alpha I^* \cos s^*(x))^{-1} \in \\ \in C^{2*}_{2\pi}(R)$$

with

$$J^* = (\alpha I^* (1 - \int_0^{2\pi} \sin^2 s^*(\xi)(I^* \cos s^*(\xi))^{-1} d\xi))^{-1} \cdot \\ \cdot \int_0^{2\pi} \wp(\xi) \cdot \operatorname{tg} s^*(\xi) d\xi =$$

$$= (\alpha I^{*2} \int_0^{2\pi} (1-2(\alpha I^*)^{-2} H^2(\xi))(1-(\alpha I^*)^{-2} H^2(\xi))^{-1/2} d\xi)^{-1} \cdot$$

$$\cdot \int_0^{2\pi} \varphi(\xi) \operatorname{tg} s^*(\xi)\, d\xi$$

(by (d) this expression has sense).

Evidently $\|\sigma\|_2 \leq C \|\varphi\|_2$. This completes the proof.

Reference

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