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ON DOMINATION PROBLEM IN BANACH ALGEBRAS

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Abstract: We give an example of a commutative Banach algebra A with elements $u, v, w \in A$ such that $|ux| \leq |vx| + |wx|$ for every $x \in A$ and there exists no commutative Banach algebra B containing A as a subalgebra and elements $b, c \in B$ such that $u = bv + cw$. This gives the negative answer to the problem of Želazko [4].

Key words: Banach algebras, ideals.

AMS: 46H05

Introduction. Let A be a unital commutative complex Banach algebra, let $u, v_1, \dots, v_n$ $(1 \leq n < \infty)$ be elements of A. As in [4] we say that u is dominated by elements $v_1, \dots, v_n$ if there exists a constant $k \geq 0$ such that $|ux| \leq k \cdot \sum_{i=1}^{n} |v_i x|$ for every $x \in A$.

Let A, B be unital commutative complex Banach algebras. We say that B is an isometric extension of A if there exists a unit preserving isometric isomorphism from A into B. In this case we consider A as a subalgebra of B and write $A \subset B$.

Let $A \subset B$, $u, v_1, \dots, v_n \in A$. Let $u = \sum_{i=1}^{n} b_i v_i$ for some $b_i \in B$. Then $|ux| \leq \sum_{i=1}^{n} |b_i| \, |v_i x| \leq k \cdot \sum_{i=1}^{n} |v_i x|$ for each $x \in A$, where $k = \max(|b_1|, \dots, |b_n|)$. So u is dominated by the elements $v_1, \dots, v_n$.

In [4] (Problem 9), the question was raised whether the converse statement is true. More precisely:

Let $u, v_1, \ldots, v_n \in A$, let u be dominated by $v_1, \ldots, v_n$. Does it follow that in some isometric extension $B \supset A$ there are elements $b_1, \ldots, b_n$ such that $u = \sum_{i=1}^{n} b_i v_i$? The answer is positive for $n = 1$ ([1]) and also for arbitrary n in special Banach algebras ([5]). In the present paper we give an example that this is not true for $n = 2$ (and of course for $n \geq 2$) in general Banach algebras.

<u>Lemma</u>. There exists a unital commutative complex Banach algebra A satisfying the following conditions:

1) There are (distinct) elements u, v, w, a_{ij} $(i,j = 0,1,2,\ldots)$ in A generating A.

2) $u^2 = v^2 = w^2 = uv = uw = vw = 0$, $a_{ij}a_{km} = 0$ for every $i, j, k, m \geq 0$

3) $a_{ij}u = a_{i-1,j}v + a_{i,j-1}w$ $(i,j \geq 1)$
   $a_{i,o}u = a_{i-1,o}v$ $(i \geq 1)$
   $a_{o,j}u = a_{o,j-1}w$ $(j \geq 1)$

4) $|a_{ij}| = 2^{-(i+j)^2}$ $(i,j \geq 0)$, $|a_{o,o}u| = 1$

5) u is dominated by v, w.

Construction: Let S be the free commutative semigroup with unit 1 and zero element 0 ($0s = 0$ for each $s \in S$) and with generators u', v', w', a'_{ij} $(i,j = 0,1,2,\ldots)$ satisfying $u'^2 = v'^2 = w'^2 = u'v' = u'w' = v'w' = 0$, $a'_{ij}a'_{km} = 0$ $(i,j,k,m = 0,1,2,\ldots)$. Put $|u'| = |v'| = |w'| = 1$, $|a'_{ij}| = |a'_{ij}u| = |a'_{ij}v| = |a'_{ij}w| = 2^{-(i+j)^2}$ for every $i, j \geq 0$.

Let B be the ℓ^1 algebra over semigroup S with this norm, i.e. B is formed by formal linear combinations

$$(1)\quad x = \lambda_0 + \lambda_1 u' + \lambda_2 v' + \lambda_3 w' + \sum_{i,j=0}^{\infty} \lambda_{ij} a'_{ij} + \sum_{i,j=0}^{\infty} \mu_{ij}^{(1)} a'_{ij} u' + \sum_{i,j=0}^{\infty} \mu_{ij}^{(2)} a'_{ij} v' + \sum_{i,j=0}^{\infty} \mu_{ij}^{(3)} a'_{ij} w'$$

where $\lambda_0, \dots, \lambda_3$, λ_{ij}, $\mu_{ij}^{(k)}$ $(k = 1,2,3,\ i,j = 0,1,\dots)$ are complex numbers and

$$|x| = \sum_{i=0}^{3} |\lambda_i| + \sum_{i,j=0}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2} + \sum_{k=1}^{3} \sum_{i,j=0}^{\infty} |\mu_{ij}^{(k)}| 2^{-(i+j)^2} < \infty .$$

Clearly B with this norm is a unital commutative Banach algebra. Let $I \subset B$ be the closed ideal generated by elements $a'_{ij}u' - a'_{i-1,j}v' - a'_{i,j-1}w'$ $(i,j \geq 1)$, $a'_{i,0}u' - a'_{i-1,0}v'$ $(i \geq 1)$ and $a'_{0,j}u' - a'_{0,j-1}w'$ $(j \geq 1)$. Denote $A = B|I$, $u = u' + I$, $v = v' + I$, $w = w' + I$, $a_{ij} = a'_{ij} + I$ $(i,j = 0,1,\dots)$. We prove that A satisfies all the conditions required.

Conditions 1), 2) and 3) are trivial.

Let us notice that if $x \in B$, x has the form (1), then

$$(2)\quad |x + I|_A = \sum_{i=0}^{3} |\lambda_i| + \sum_{i,j=0}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2} + |\mu_{00}^{(1)}| + \sum_{i+j\geq 1} \inf_{\nu \in C} |\mu_{ij}^{(1)} a'_{ij} u' + \mu_{i-1,j}^{(2)} a'_{i-1,j} v' + \mu_{i,j-1}^{(3)} a'_{i,j-1} w' + \nu a'_{ij} u' - \nu a'_{i-1,j} v' - \nu a'_{i,j-1} w'|_B = \sum_{i=0}^{3} |\lambda_i| + \sum_{i,j=0}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2} + \sum_{i,j=0}^{\infty} |\mu_{ij}^{(1)} a_{ij} u + \mu_{i-1,j}^{(2)} a_{i-1,j} v + \mu_{i,j-1}^{(3)} a_{i,j-1} w|_A .$$

(Here we put $a_{km} = a'_{km} = 0$ for $\min(k,m) < 0$.)

From formula (2), the condition 4) immediately follows. Further, it holds

$$(3) \qquad |a_{ij}u| = 2^{-(i+j)^2} \quad (i,j \ge 0)$$

$$|a_{ij}v| = 2^{-(i+j)^2} \quad (i \ge 0,\ j \ge 1)$$

$$|a_{ij}w| = 2^{-(i+j)^2} \quad (i \ge 1,\ j \ge 0)$$

$$|a_{i,o}v| = |a_{i+1,o}u| = 2^{-(i+1)^2} \quad (i \ge 0)$$

$$|a_{o,j}w| = |a_{o,j+1}u| = 2^{-(j+1)^2} \quad (j \ge 0).$$

It remains to prove the condition 5). Let $x \in B$ have the form (1), $y = x + I \in A$. Then by (2),(3)

$$|yu_A| = |xu + I|_A = |\lambda_o u' + \sum_{i,j=0}^{\infty} \lambda_{ij} a'_{ij} u' + I|_A = |\lambda_o| + \sum_{i,j=0}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2},$$

$$|yv|_A = |xv' + I|_A = |\lambda_o v' + \sum_{i,j=0}^{\infty} \lambda_{ij} a'_{ij} v' + I|_A = |\lambda_o| + \sum_{i=0}^{\infty} \sum_{i=1}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2} + \sum_{i=0}^{\infty} |\lambda_{i,o}| 2^{-(i+1)^2},$$

$$|yw|_A = |xw' + I|_A = |\lambda_o w' + \sum_{i,j=0} \lambda_{ij} a'_{ij} w' + I|_A = |\lambda_o| + \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} |\lambda_{ij}| 2^{-(i+j)^2} + \sum_{j=0}^{\infty} |\lambda_{o,j}| 2^{-(j+1)^2}.$$

From this immediately follows $|yu| \le |yv| + |yw|$ for every $y \in A$, hence u is dominated by v, w. Note that u is not dominated by v. It is $|a_{i,o}u|/|a_{i,o}v| = 2^{-i^2}/2^{-(i+1)^2} = 2^{2i+1}$ which forms an unbounded sequence. Similarly, neither u is dominated by w.

<u>Theorem</u>. Let A be the Banach algebra from the previous Lemma. Let C be any isometric extension of A. Then there

exist no b, $c \in C$ such that $u = bv + cw$.

<u>Proof</u>: I. Let $k > 0$ be fixed. Let B_k be the ℓ^1 algebra over the free commutative semigroup with generators b_k, c_k with coefficients in A, i.e. B_k consists of elements of the form $x = \sum_{i,j=0}^{\infty} x_{ij} b_k^i c_k^j$, where $x_{ij} \in A$ $(i,j \geq 0)$ and $|x|_{B_k} = \sum_{i,j=0}^{\infty} |x_{ij}|_A \cdot k^{i+j}$.

Algebraic operations in B_k are defined as follows:

For $y = \sum_{i,j=0}^{\infty} y_{ij} b_k^i c_k^j$ it is $x + y = \sum_{i,j=0}^{\infty} (x_{ij} + y_{ij}) b_k^i c_k^j$,

$xy = yx = \sum_{m,n=0}^{\infty} b_k^m c_k^n \left(\sum_{\substack{i+i'=m \\ j+j'=n}} x_{ij} y_{i'j'} \right)$.

Clearly B_k is a Banach algebra, $B_k \supset A$. Denote $z = u - b_k v - c_k w$. Let $J = \overline{zB_k}$ be the closed ideal generated by z. Denote $d = \sum_{i,j=0}^{\infty} a_{ij} b_k^i c_k^j$ where a_{ij} are elements from the previous Lemma. It holds

$$|d|_{B_k} = \sum_{i,j=0}^{\infty} |a_{ij}|_A \cdot k^{i+j} = \sum_{i,j=0}^{\infty} 2^{-(i+j)^2} \cdot k^{i+j} = \sum_{m=0}^{\infty} 2^{-m^2} \cdot k^m (m+1) < \infty$$

. So $d \in B_k$. We have

$$dz = \left(\sum_{i,j=0}^{\infty} a_{ij} b_k^i c_k^j\right)(u - b_k v - c_k q) = a_{0,0} u + \sum_{j=1}^{\infty} b_k^i (a_{i,0} u - a_{i-1,0} v) + \sum_{j=1}^{\infty} c_k^j (a_{0,j} u - a_{0,j-1} q) + \sum_{i,j=1}^{\infty} b_k^i c_k^j (a_{ij} u - a_{i-1,j} v - a_{i,j-1} w) = a_{0,0} u.$$

Hence $a_{0,0} u \in J$.

II. Suppose now on the contrary that there exists a Banach algebra C containing A as a subalgebra and b, $c \in C$ such that $u = bv + cw$. Choose $k \geq \max(|b|, |c|)$. Define a homomorphism $f: B_k \longrightarrow C$ by $f\left(\sum_{i,j=0}^{\infty} x_{ij} b_k^i c_k^j\right) = \sum_{i,j=0}^{\infty} x_{ij} b^i c^j$. It is

$$\left|\sum_{i,j=0}^{\infty} x_{ij} b^i c^j\right|_C \leq \sum_{i,j=0}^{\infty} |x_{ij}|_A |b|_C^i \; |c|_C^j \leq \sum_{i,j=0}^{\infty} |x_{ij}|_A \cdot k^{i+j} = \left|\sum_{i,j=0}^{\infty} x_{ij} b_k^i c_k^j\right|_{B_k}.$$

So the definition of f is correct and $|f| \leq 1$. Clearly

$f(b_k) = b$, $f(c_k) = c$ and $f|A$ is the identical mapping (we identify elements of A with the corresponding elements of B_k and C, respectively). It holds $f(z) = f(u - b_k v - c_k w) = u - bv - cw = 0$, so $f(J) = 0$. Hence $f(a_{o,o}u) = 0$. On the other hand, $a_{o,o}u \in A$ and $f|A$ is the identical mapping. Necessarily $a_{o,o}u = 0$ which contradicts the condition 4) of Lemma.

Remark 1: A Banach algebra B is called an extension of a Banach algebra A if there exists a unit preserving topological isomorphism of A into B. It is easy to see that the words "isometric extension" in the Theorem can be replaced by "extension". The proof in this case is the same. Note also that every extension C of A becomes an isometric extension after a suitable renorming of C (see [2]).

Remark 2: The following question still remains open: Let 1 (unit element of A) be dominated by $v_1, \dots, v_n \in A$. Does it follow that $1 = \sum_{i=1}^{n} b_i v_i$ for some extension B of A and some $b_i \in B$?

This question is equivalent to Problem 5 of [4]: Does every non-removable ideal in A consist of joint topological divisors of zero?

For related topics see also [3].

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