Andrzej Szymański
Some Baire category type theorems for $U(\omega_1)$

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Abstract: It is shown that if $\omega \omega$ has an $\omega_1$-scale, then $U(\omega_1)$ can be covered by $\omega_1$ $G_\delta$ closed and nowhere dense subsets of $U(\omega_1)$ and that the union of countably many of them is dense in $U(\omega_1)$. On the other hand, we show that under MA+$\neg$CH, the union of countably many $G_\delta$, closed and nowhere dense subsets of $U(\omega_1)$ is nowhere dense in $U(\omega_1)$. For these purposes we use the notion of $k$-matrices on $\omega_1$.

Key words and phrases: Ultrafilter, uniform ultrafilter, matrix, scale.

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In this note we consider families consisting of $G$ closed and nowhere dense subsets of $U(\omega_1)$. We are mainly interested in the question, what cardinalities have such families, as above, which cover $U(\omega_1)$ or have a dense union. Some results in this direction are obtained. For example, it is shown (Theorem 2) that if $\omega \omega$ has an $\omega_1$-scale, then such a family of cardinality $\omega_1$ exists which covers $U(\omega_1)$ and, in addition, it contains a countable subfamily with a dense union. The same conclusions have been obtained by Balcar and Vopěnka [BV] when...
\( \omega_1 = \omega_2 \) holds, however, without possibility to get \( G_{\mathcal{F}} \)-sets. Our result also shows that if \( \omega_1 \) has an \( \omega_1 \)-scale, then the Novák number of \( U(\omega_1) \), \( n(U(\omega_1)) \), is \( \not\in \omega_1 \). Recall [KS] that the Novák number of a dense in itself topological space \( X \), \( n(X) \), is the minimal cardinality of a family consisting of nowhere dense sets covering the whole space. For the short history concerning the Novák number of various topological spaces, we refer to [BPS].

The existence of families consisting of \( G \) closed and nowhere dense subsets of \( U(\omega_1) \) is closely related to the existence of \( \kappa \)-matrices on \( \omega_1 \), as is shown in Theorem 4, and the existence of \( \kappa \)-matrices on \( \omega_1 \) for \( \kappa \geq \omega_1 \) is related to the question whether \( \beta \omega_1 - \omega_1 \) is homeomorphic to \( \beta \omega - \omega \) (Theorem 6).

All of the above results are independent of the ZFC axioms since if \( Q \) holds, then the union of countably many \( G_{\mathcal{F}} \)-closed and nowhere dense subsets of \( U(\omega_1) \) is nowhere dense in \( U(\omega_1) \) (Theorem 8).

**Conventions and notations.** As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Cardinals carry the discrete topology. If \( A, B \) are sets, then \( A B \) is the set of all functions from \( A \) into \( B \). If \( \varphi, \psi \in \omega_\omega \), then \( \varphi \not< \psi \) means that \( |\{ n : \varphi(n) \not< \psi(n) \}| < \omega \). A subset \( F \subset \omega_\omega \) is dominant if for every \( \varphi \in \omega_\omega \) there is a \( \psi \in F \) such that \( \varphi \not< \psi \). A scale is a well ordered by \( \not< \) increasing dominating family. If \( \kappa \) is a cardinal and \( A, B \subset \kappa \), then \( A \) and \( B \) are almost disjoint if \( |A| = \kappa = |B| \) and \( |A \cap B| < \kappa \). We denote by \( U(\omega_1) \) the space of uniform ultrafilters on \( \omega_1 \).
Results. We begin from the following simple

Lemma 1. A set $F \subseteq U(\omega_1)$ is $\mathcal{G}_\delta$ closed and nowhere dense in $U(\omega_1)$ iff for any sets $A_n \subseteq \omega_1$, $n < \omega$, such that $F = \bigcap \{ \beta \in \omega_1 : m_n < \omega \} \cap U(\omega_1)$ there is $\bigcap \{ A_n : m_n < \omega \} \subseteq \omega$ if there are sets $B_n \subseteq \omega_1$ such that $F = \bigcap \{ \beta \in \omega_1 : m_n < \omega \} \cap U(\omega_1)$, $B_1 \supseteq B_2 \supseteq \ldots$ and $\bigcap \{ B_n : m_n < \omega \} = \emptyset$.

Theorem 2. If $\omega_\omega$ has an $\omega_1$-scale, then $U(\omega_1)$ can be covered by $\mathcal{G}_\delta$ closed and nowhere dense subsets of $U(\omega_1)$.

In particular, if $\omega_\omega$ has an $\omega_1$-scale, then $\mu(U(\omega_1)) = \omega_1$.

Proof. Let $\{ \varphi_\omega : \alpha < \omega_1 \}$ be an $\omega_1$-scale in $\omega_\omega$. For each $n, m < \omega$ we set $\lambda^n_m = \{ \alpha : \varphi_\omega(m) \in \alpha \}$. Observe that:

0) if $m < k < \omega$ and $n < \omega$, then $\lambda^n_m \subseteq \lambda^n_k$;

(i) $U \{ \lambda^n_m : m < \omega \} = \omega_1$ for each $n < \omega$;

(ii) for each infinite $s < \omega$ and $\psi \in s^\omega$, $\bigcap \{ \lambda^n_m : m \in s \} \subseteq \omega$.

The properties of $\lambda^n_m$'s stated in (0) and (i) are obvious.

For the proof of (ii) let us assume on the contrary that $\bigcap \{ \lambda^n_m : m \in s \} > \omega$ for some infinite $s < \omega$ and $\psi \in s_\omega$.

There exists an $\alpha < \omega_1$ such that $\varphi_\omega(s \in \psi)$ $\alpha$. Since $\bigcap \{ \lambda^n_m : m \in s \}$ is uncountable, there exists a $\beta \in \bigcap \{ \lambda^n_m : m \in s \}$ such that $\omega_1 > \beta > \alpha$. Since $\{ \varphi_\omega : \alpha < \omega_1 \}$ is a scale, $\varphi_\omega(\beta) \nsubseteq \varphi_\omega$. Hence there is an $n \in s$ such that $\varphi_\omega(n) > \psi(n)$.

But this means that $\beta \notin \lambda^n_m$; a contradiction.

Now define the sets $F_n$ and $E_n$ in the following way:

$F_n = \{ \xi \in U(\omega_1) : \lambda^n_m(\xi) \}\subseteq \{ \xi \in U(\omega_1) : \varphi_\omega(n) \}\subseteq \{ \xi \in U(\omega_1) : \varphi_\omega(n) \}

for each $m < \omega$ and

$E_n = \{ \xi \in U(\omega_1) : \lambda^n_m(\xi) \}\subseteq \{ \xi \in U(\omega_1) : \varphi_\omega(n) \}\subseteq \{ \xi \in U(\omega_1) : \varphi_\omega(n) \}

for each $m \geq n$.

In the topological language, $F_n = \bigcap \{ \varphi_\omega(\omega_1 - \lambda^n_m) : m < \omega \}$.
and \( E^\alpha = \bigcap \{ \alpha \downarrow \omega \downarrow \alpha \downarrow \alpha \} : m \geq n \}. \) Of course \( F_n \) as well as \( E^\omega \) are \( G_\gamma \) closed subsets of \( U(\omega) \). From (i) and Lemma 1 it follows that \( F_n \) is nowhere dense in \( U(\omega) \) for each \( n < \omega \), and from (ii) and Lemma 1 it follows that \( E^\omega \) is nowhere dense in \( U(\omega) \) for each \( n < \omega \) and \( \alpha < \omega \). It remains to show that \( \bigcup \{ F_n : n < \omega \} \cup \{ E^\alpha : \alpha < \omega \} = U(\omega) \). For this, let \( \xi \in U(\omega) \) be such that \( \xi \notin \bigcup \{ F_n : n < \omega \} \). From (i) it follows that for each \( n < \omega \) there exists \( \psi(n) < \omega \) such that \( A^\psi(n) \in \xi \). Let \( \alpha < \omega \) be such that \( \exists n \geq \alpha \psi(n) \). This means that there exists an \( m < \omega \) such that \( \exists \psi(n) > \xi \) for each \( n \geq m \). Hence \( \xi \in E^\omega \).

The above theorem is related to a result by Balcar and Vopěnka [BV] who proved that if \( 2^{\omega} = \omega_2 \), then \( n(U(\omega)) = \omega_1 \). However, the following consistency results are known:

( \( \omega \), \( \omega_1 \)-scale + \( \omega_1 = 2^\omega + 2^\omega \) arbitrarily large) [H],

(\( \omega \), \( \omega_1 \)-scale + \( \omega_1 + \omega = \omega \) [MS]),

(\( \omega \), \( \omega_1 \)-scale + \( \omega_1 = \omega_2 \) (a model for Martin's axiom + \( 2^\omega = \omega_2 \))),

(\( \omega \), \( \omega_1 \)-scale + \( \omega_1 = \omega_2 \) (a model for GCH).

In the proof of the Theorem 2, we have constructed a matrix \( \{ A^m_n : m, n < \omega \} \) satisfying conditions (0),(i),(ii). Now we generalize this notion by saying that a matrix \( \{ A^m_n : m, n < \omega \} \) of subsets of \( \omega_1 \) is a \( \kappa \)-matrix on \( \omega_1 \) if the following hold:

(0) if \( m < n \) and \( \alpha < \kappa \), then \( \alpha \downarrow A^m_n \subset A^m_n \).

(i) \( \bigcup \{ A^m_n : n < \omega \} = \omega \) for each \( \alpha < \kappa \),

(ii) for each infinite \( s < \kappa \) and \( \psi \in \mathcal{P}(\omega) \), \( \bigcap \{ A^\psi(\alpha) : \alpha \in s \} \leq \omega \).

Thus we have shown
Proposition 3. If \( \omega \) has an \( \omega_1 \)-scale, then there exists an \( \omega \)-matrix on \( \omega_1 \).

Now we shall give a topological reformulation of the existence of \( \kappa \)-matrices on \( \omega_1 \).

Theorem 4. A \( \kappa \)-matrix on \( \omega_1 \) exists iff there exists a family consisting of at least \( \kappa \) \( G_{\rho} \)-closed and nowhere dense subsets of \( U(\omega_1) \) such that each union of infinitely many of them is dense in \( U(\omega_1) \).

Proof. Assume \( \{ A_{\alpha} \} : n < \omega, \alpha < \kappa \} \) is a \( \kappa \)-matrix on \( \omega_1 \). For \( \alpha < \kappa \) we put \( F_{\alpha} = \{ f \in U(\omega_1): A_{\alpha} \not\supset f \} \) for each \( n < \omega \). Obviously, each \( F_{\alpha} \) is a \( G_{\rho} \)-closed and nowhere dense subset of \( U(\omega_1) \), in virtue of Lemma 1 and (i). Choose infinitely many of them, say \( F_{\alpha_1}, F_{\alpha_2}, \ldots \) and assume on the contrary that \( \cup F_{\alpha_1} \cup F_{\alpha_2} \cup \ldots \) is not dense in \( U(\omega_1) \). This means that there exists an uncountable set \( B \subseteq \omega_1 \) such that \( \text{cl}_{\omega_1} B \cap F_{\alpha} = \emptyset \) for each \( n < \omega \). Hence, by (0) and (i), for each \( n < \omega \) there exists a \( \gamma_n \times \omega \) such that \( | B - A_{\alpha_n}^n | = \omega \). Hence \( B \) contains an uncountable subset \( C \) such that \( C \subseteq A_{\alpha_n}^n \) for each \( n < \omega \). But then, for some infinite set \( s = \{ \alpha_1, \alpha_2, \ldots \} \) contained in \( \kappa \) and a \( \gamma \in \alpha_\omega \) given by \( \gamma(\alpha_n) = \gamma_n \), we have \( | \cap \{ A(\gamma) : \alpha \in s \} | \geq Z | C | = \omega_1 \), which contradicts (ii).

Let \( F_{\alpha}, \alpha < \kappa \), be \( G_{\rho} \)-closed and nowhere dense subsets of \( U(\omega_1) \) such that each union of infinitely many of them is dense in \( U(\omega_1) \). By Lemma 1, for each \( \alpha < \kappa \) there are sets \( B_{n}^{\alpha}, n < \omega \), such that \( F_{\alpha} = \cap \{ \text{cl}_{\omega_1} B_{n}^{\alpha} \cap U(\omega_1): n < \omega \} \), \( B_{n}^{\alpha} \supseteq \supseteq B_{n}^{\alpha} \supseteq \ldots \) and \( \cap \{ B_{n}^{\alpha}: n < \omega \} = \emptyset \). Setting \( A^{\alpha} = \omega_1 - B_{n}^{\alpha} \) we see that the matrix \( \{ A_{n}^{\alpha} : n < \omega, \alpha < \kappa \} \) fulfills conditions (0) and (i). We verify (ii). Choose an arbitrary infinite set \( s \subset \kappa \)
and \( \forall \in \omega_1 \). By the assumption, \( \bigcup \{ F_\alpha : \alpha \in \omega \} \) is dense in \( U(\omega_1) \), so that \( \bigcap \{ \alpha \omega_1 \wedge (\forall \alpha) \cap U(\omega_1) : \alpha \in \omega \} \) is nowhere dense in \( U(\omega_1) \). Hence, by Lemma 1, \( \bigcap \{ A^{(\forall \alpha)} : \alpha \in \omega \} \not\subseteq \omega_1 \).

**Corollary 5.** An \( \omega \)-matrix on \( \omega_1 \) exists if there is a countable family \( F \) consisting of \( G_\sigma \)-closed and nowhere dense subsets of \( U(\omega_1) \) such that \( \bigcup F \) is dense in \( U(\omega_1) \).

**Proof.** If \( F = \{ F_n : n < \omega \} \), then letting \( F_1 = E_1 \) and \( F_n = E_1 \cup E_2 \cup \ldots \cup E_n \) for \( 1 < n < \omega \), we see that each \( E_n \) is a \( G_\sigma \)-closed and nowhere dense subset of \( U(\omega_1) \) such that each union of infinitely many of them is dense in \( U(\omega_1) \), since it is equal to \( \bigcup F \).

The above topological equivalence of the existence of \( \kappa \)-matrices on \( \omega_1 \) seems to be rather pathological, for \( \kappa \geq \omega_1 \). For example, it cannot happen in topological spaces which have a pseudobase of cardinality less than \( \kappa \). However, we have

**Theorem 6.** If \( \beta \omega_1 - \omega_1 \) is homeomorphic to \( \beta \omega - \omega \) and there exists an almost disjoint family on \( \omega_1 \) of cardinality \( \kappa \), then there exists a \( \kappa \)-matrix on \( \omega_1 \).

**Proof.** Decompose \( \omega_1 \) into \( \omega_1 \)-disjoint subsets \( B_\omega \) of cardinality \( \omega_1 \), say \( B_\omega = \{ b_\beta : \beta < \omega_1 \} \). Let \( F = \{ f_\beta : \beta < \kappa \} \) be a family consisting of almost disjoint subsets of \( \omega_1 \). Let \( \varphi_\beta \) be an isomorphism between \( \omega_1 \) and a well ordered set \( \omega_\beta \). Then we put \( C_\beta = \{ b_\alpha^{(\varphi_\beta)} : \alpha < \omega_\beta \} \). Note that sets \( C_\beta \) defined in such a way are also almost disjoint and \( |C_\beta \cap B_\omega| = 1 \) for each \( \beta < \kappa \) and \( \alpha < \omega_1 \).

Let \( \phi \) be a Boolean isomorphism between the Boolean al-
gebras $P(\omega_1)/\text{mod fin}$ and $P(\omega)/\text{mod fin}$. Choose $B_\alpha^\prime \in \phi([B_\alpha])$ and $C_\xi^\prime \in \phi([C_\xi])$. Then we define $A_\xi^n = \{ \alpha : B_\alpha^\prime \cap C_\xi^\prime \subseteq n \}$. The matrix $\{ A_\xi^n : \xi < \kappa, n < \omega \}$ is a $\kappa$-matrix on $\omega_1$. To see this, observe that conditions (0) and (i) follow from the fact that $B_\alpha^\prime$ and $C_\xi^\prime$ are almost disjoint subsets of $\omega_1$ for each $\alpha < \omega_1$ and $\xi < \kappa$. We verify (ii). Let infinite $s \subseteq \kappa$ and $s \subseteq \omega$ be given. Assume on the contrary that $| \cap \{ A_\xi^n : \xi \in s \} | > \omega$. Without loss of generality we may assume that $s$ is countable. Let $D' = \bigcup \{ C_\xi^\prime - \psi(\xi) : \xi \in s \}$ and choose $D \in \phi^{-1}(D')$. Since $| C_\xi^\prime - D' | < \omega$ for each $\xi \in s$, $| C_\xi - D | < \omega$ for each $\xi \in s$. Since $s$ is countable, there is a $\beta < \omega_1$ such that $C_\xi^\prime - \bigcup \{ B_\alpha : \alpha < \beta \} \subseteq D$. Since the sets $C_\xi^\prime$ are almost disjoint, there is a $\gamma < \omega_1$ such that $C_\xi^\prime - \bigcup \{ B_\alpha : \alpha < \gamma \}$ are disjoint for each $\xi \in s$. Consequently, $| \bigcup \{ C_\xi^\prime : \xi \in s \} \cap B_\alpha | = \omega$ for each $\alpha > \gamma$. Choose $\eta \in \cap \{ A_\xi^n : \xi \in s \}$ such that $\eta > \beta$ and $\eta > \gamma$. Then $| B_\eta \cap D | = \omega$ and therefore $| B_\eta \cap D' | = \omega$, too. Thus $B_\eta^\prime \cap C_\xi^\prime \not\subseteq \psi(\xi)$ for infinitely many $\xi$. Hence $\eta \not\in \cap \{ A_\xi^n : \xi \in s \}$, a contradiction.

Since there exists always an almost disjoint family on $\omega_1$ of cardinality $\omega_2$, we have

Corollary 7. If $\beta \omega_1 - \omega_1$ is homeomorphic to $\beta \omega - \omega$, then there exists an $\omega_2$-matrix on $\omega_1$.

The problem to distinguish topologically the spaces $\omega_1^\beta - \omega_1$ and $\beta \omega - \omega$ is not yet solved; for partial solutions see [F], [BF].

Some theorems above show what kinds of conditions allow to get the existence of some $\kappa$-matrices on $\omega_1$. The next
theorem refutes such a possibility.

Q means that if \( F \subset \omega^\omega \) and \(|F| \leq \omega_1\), then there is a \( \nu \in \omega^\omega \) such that \( \varphi \leq \nu \) for each \( \varphi \in F \).

**Theorem 8.** If Q, then there is no \( \omega \)-matrix on \( \omega_1 \).

Proof. Assume otherwise and let \( \{A^n_m:n,m<\omega\} \) be an \( \omega \)-matrix on \( \omega_1 \). For \( \varphi \in \omega^\omega \) we let \( a^\varphi = \sup \{b^\varphi_m:n<\omega\} \), where \( b^\varphi_n = \sup \{A^\varphi_k:k \geq n\} \). Since \( \{A^n_m:n,m<\omega\} \) is an \( \omega \)-matrix on \( \omega_1 \), \( a^\varphi < \omega_1 \) for each \( \varphi \in \omega^\omega \). Now, we claim that for each \( \alpha < \omega_1 \) there is a \( \varphi_{\omega} < \omega_\omega \) such that \( \varphi_{\omega} < \alpha \). To see this, we note that from condition (i) for \( \kappa \)-matrices it follows that for each \( n < \omega \) there exists \( \varphi_n < \omega \) such that \( \alpha \in A^n_n \). So, taking \( \varphi_{\omega} \) such that \( \varphi_{\omega}(n) = \varphi_n \), we have \( \alpha_{\omega} \geq \omega \). By Q, there exists a \( \nu \in \omega^\omega \) such that \( \varphi_{\omega} < \nu \) for each \( \alpha < \omega_1 \). Let \( \beta < \omega_1 \). Since \( \varphi_{\beta} \neq \nu \), there exists an \( n < \omega \) such that \( \varphi_{\beta}(k) < \nu(k) \) for \( k \geq n \). Then, by (0) for \( \kappa \)-matrices, \( A_{\beta}^\varphi(k) \subset A_{\nu}^\varphi(k) \) for \( k \geq n \). Hence \( \cap \{A_{\beta}^\varphi(k):k \geq n\} \subset \{A_{\nu}^\varphi(k):k \geq n\} \), and therefore \( b_{\beta}^\varphi_m \leq b_{\nu}^\varphi_m \) for each \( m \geq n \). In consequence, \( \beta \leq a_{\beta}^\varphi = \sup \{b_{\beta}^\varphi_m:n<\omega\} \leq \sup \{b_{\nu}^\varphi_m:n<\omega\} = a_{\nu}^\varphi \). Hence \( a_{\nu}^\varphi = \omega_1 \); a contradiction.

It is well known that Martin's axiom + \( \neg \)CH implies Q ([MS]). So we have

**Corollary 9 (MA + \( \neg \)CH).** If \( F \) is a countable family consisting of \( G \) closed and nowhere dense subsets of \( U(\omega_1) \), then \( \bigcup F \) is nowhere dense in \( U(\omega_1) \).

If \( F \) is a countable family consisting of disjoint closed and nowhere dense subsets of \( U(\omega_1) \), then \( \bigcup F \) is nowhere dense in \( U(\omega_1) \).

Proof. Assume otherwise. Then, by Corollary 5, some un-
countable subset of \( \omega_1 \) would have an \( \omega \)-matrix. But this contradicts Theorem 8.

The second part of the corollary follows immediately from the first part.

It may be worthwhile to point out that the assumptions on the family \( F \) in Corollary 9 are essential, since Balcar and Vopěnka \([BV]\) showed that if \( 2^{\omega_1} = \omega_2 \), then there exists a countable family \( F' \) consisting of closed and nowhere dense subsets of \( U(\omega_1) \) such that \( \bigcup F' \) is dense in \( U(\omega_1) \). Also \( 2^{\omega_1} = \omega_2 \) is consistent with MA + \( \neg \text{CH} \).

**Question.** Does the existence of \( \kappa \)-matrices on \( \omega_1 \), for \( \kappa \geq \omega_1 \), be consistent with ZFC?

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Uniwersytet Śląski
Instytut Matematyki
Bankowa 14, 40-007 Katowice
POLSKA

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