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CLOSURE OPERATIONS AND SEMIGROUPS OF QUOTIENTS

John K. LUEDEMAN

Abstract: The semigroup of quotients of a monoid $S$ with 0 is constructed by means of a closure operation $c$ on the lattice of left ideals of $S$. The lattice of closure operations on $L$ is isomorphic to the lattice of left quotient filters $\mathcal{L}_0$ on $S$. These closure operations enable one to describe Green's relation $L$ on $Q(S)$.

Key words: Semigroup of quotients, injective hull, closure operation.

Classification: 20 M 50

The discussion of semigroups of quotients has been modelled on the discussion of rings of quotients. One useful way of discussing rings of quotients was by the use of closure operations on the lattice of left ideals of a ring [2]. In this paper, we give a generalization of the work of Murdoch [2] on rings to semigroups. This generalization gives yet another way to describe the maximal semigroups of quotients of $S$. This new description leads to a characterization of Green's relation $L_Q$ on $Q(S)$ and shows the connection between $L_S$ on $S$ and $L_Q$ on $Q(S)$.

A nonempty collection $\mathcal{L}$ of left ideals of $S$ is a left quotient filter if
Let $L$ be the collection of left ideals of $S$. A closure operation is a mapping $A \rightarrow A^c$ of $L$ into itself satisfying

1. $A \subseteq A^c$
2. $A \subseteq B \Rightarrow A^c \subseteq B^c$
3. $A^{cc} = A^c$
4. If $f \in \text{Hom}_S(S,S)$, then $f^{-1}(A^c) = (f^{-1}A)^c$.

An element is c-closed if $A^c = A$, and is c-dense if...
\( A^c = S \). The intersection of any collection of \( c \)-closed elements is \( c \)-closed and so the closure \( A^c \) of \( A \) is the intersection of all \( c \)-closed elements containing \( A \). Thus the closure operation is completely determined by the lattice \( L^c \) of closed elements of \( L \), and \( L^c \) is an inset in the sense that \( S \in L^c \), \( L^c \subseteq L \) and \( L^c \) is closed under complete intersections. Conversely, each inset determines a unique closure operation on \( L \).

Given a left quotient filter \( \Sigma \) on \( S \), let \( A \in L \) and set \( A^c = \{ s \in S | As^{-1} \in \Sigma \} \). Then the mapping \( \Lambda \rightarrow \Lambda^c \) is a closure operation on \( L \). Note that (2) follows from (6.1), (3) follows from (6.3) and (4) follows from (6.2).

Conversely, given a closure operation \( \Lambda \rightarrow \Lambda^c \) on \( L \), let \( \Sigma = \{ A \in L | \Lambda^c = S \} \).

**Proposition:** \( \Sigma \) is a left quotient filter.

**Proof:** Condition (6.1) follows from (2). Condition (6.2) follows immediately from (4). In order to verify (6.3), let \( \Lambda \in \Sigma \) and \( Ba^{-1} \in \Sigma \) for all \( a \in A \). Then \( \bigcup_{a \in A} (Ba^{-1}) a \subseteq B \), and so \( \bigcup_{a \in A} (Ba^{-1}) a^c \supseteq \bigcup_{a \in A} [(Ba^{-1}) a]^c \). Define \( f_a : S \rightarrow S \) by \( f_a(s) = sa \). Then by (3), \( (Ba^{-1}) a^c = Sa \subseteq (Ba^{-1}) a \) so \( \bigcup_{a \in A} Sa = A \subseteq \bigcup (Ba^{-1}) a \subseteq \bigcup [(Ba^{-1}) a]^c \subseteq B^c \) so \( \Lambda^c = S \subseteq B^c \) and \( B \in \Sigma \).

Given a left quotient filter \( \Sigma \), we obtain a closure operation \( \Lambda \rightarrow \Lambda^c \) and from this a left quotient filter \( \Sigma^c \).

**Proposition:** \( \Sigma = \Sigma^c \).

**Proof:** Let \( \Lambda \in \Sigma^c \), then \( \Lambda^c = S \) so for all \( s \in S \), there is \( B_s \in \Sigma \) with \( B_s s \subseteq A \). Thus \( S \subseteq \Sigma \) and \( As^{-1} \in \Sigma \) for all \( s \in S \) implies \( A \in \Sigma \) by (6.3). Conversely, \( A \in \Sigma \) implies \( As^{-1} \in \Sigma \) for all \( s \in S \) so \( A^c = S \) or \( A \in \Sigma^c \).
Conversely, let \((\ )^c\) be a closure operation on \(L\), and let \((\ )^{c_1}\) be the closure operation given by \(\Sigma^c\).

**Proposition:** \(c = c^{1}\).

**Proof:** We have \(y \in B^{c_1}\) iff \(Sy \subseteq B^{c_1}\) iff \(S \subseteq B^{c_1}y^{-1} = (By^{-1})^{c_1}\) iff \((By^{-1})^{c_1} = S\) iff \((By^{-1})^{c} = S\) iff \(B^{c}y^{-1} = S\) iff \(y \in B^{c}\).

Define an order on the class of closure operations on \(L\) by \(a \preceq b\) iff \(L^a \subseteq L^b\). Thus if \(a \preceq b\), for \(A \in L\) we have \(A^b \subseteq A^a\).

Now \(L^a \cap L^b\) is an inset and so yields a closure operation \(a \triangleleft b\). Likewise \(L^a \cup L^b\), the set consisting of all elements of the form \(A \cap B\) where \(A \in L^a\) and \(B \in L^b\), is also an inset and so yields a closure operation \(a \circ b\). This gives the class of closure operations on \(L\) the structure of a complete lattice.

Combining these results we have the following:

**Theorem:** The collection of left quotient filters on \(S\) is in one-to-one correspondence with the lattice of closure operations on \(L\) by the map \(\Sigma \mapsto (\ )^c\). Moreover, this map satisfies \(\Sigma_{a \circ b} = \Sigma_a \cap \Sigma_b\).

We remark that each closure operation can be extended to the category of left \(S\)-systems by defining for a left \(S\)-system \(M, E(M)\) the injective hull of \(M\), and

\[
M^c = \{e \in E(M) : Me^{-1} \in \Sigma\}.
\]

Thus to each closure operation \(c\) there corresponds a torsion theory \((T, F)\) given by \(T = \{A | O^c = A\}\) and \(F = \{B | O^c = 0\}\), where by \(O^c = A\) we mean that for \(a \in A\) there is \(T \in \Sigma\) with \(Ta = 0\). We refer the interested reader to [1] for a discussion of torsion theories.
2. **Closure Operations and Injective Hulls.** Let $c$ be a closure operation on $L$ and let $\Xi$ be its corresponding left quotient filter. Let $g^M$ be a strongly torsion free left $S$-system. Let $E(M) = E$ be the injective hull of $M$. Let $T$ be the closure of $M$ in $E$, that is, $T = \{ e \in E | \exists A \in M \text{ for some } A \in \Xi \}$. Now $M$ is $\cap$-large in $T$ since $O^c = 0$.

**Lemma:** $T$ is $\Xi$-injective.

**Proof:** Let

$$\begin{array}{c}
A \subseteq B \\
\downarrow f \downarrow g \\
T \subseteq E
\end{array}$$

where $A \subseteq' B$ means that for all $b \in B$, there is $C \in \Xi$ with $Cb \subseteq A$. Then if $g(b) \notin T$, there is $C \in \Xi$ with $Cb \subseteq A$ so $g(Cb) = g(Cb) \subseteq f(A)$, so $g(b) \notin T^c = M^{cc} = M^c = T$.

Thus $T$ is a $\Xi$-injective $\cap$-large extension of $M$. $T$ is unique up to isomorphism over $M$ for if $F$ is another $\Xi$-injective $\cap$-large extension of $M$, then we have the commutative diagram

$$\begin{array}{c}
M \subseteq T \\
\downarrow h \downarrow g \\
M \subseteq F
\end{array}$$

Then $g = h$ if we can show the following

**Lemma:** $T$ is strongly $\Xi$-injective.

**Proof:** Let

$$\begin{array}{c}
A \subseteq' B \\
\downarrow f \downarrow h \downarrow g \\
T \subseteq E
\end{array}$$
be as before and suppose for some \( b \in B \), \( g(b) \neq h(b) \). Then for \( C \in \Sigma \) with \( Cb \subseteq A \), \( Cg(b) = Ch(b) \). Thus \( (g(b), h(b)) \in \tau_T = \text{id} \) so \( g(b) = h(b) \).

In [1], we showed that when \( M = S, T \) is a semigroup and is called the \( \Sigma \)-semigroup of quotients of \( S \) and is denoted by \( Q_\Sigma (S) \). We next examine whether the subsemigroups of \( Q(S) \), the maximal Utumi semigroup of quotients of \( S \), all occur as \( Q_\Sigma (S) \) for some left quotient filter \( \Sigma \). Since each left quotient filter is determined by a unique closure operation \( c \), we will discuss subrings of \( Q(S) \) corresponding to bilateral closure operations on \( L \). A closure operation \( c \) on \( L \) is bilateral if \( A^c \) is a two-sided ideal.

Recall the construction of \( Q(S) \) as \( Q(S) = B/\Theta \) where \( B = \bigcup_{A \in \Sigma} \text{Hom} (A, S) \) with operation \( fg \) as composition where \( f: I_f \to S, g: I_g \to S \) then \( fg: I_{fg} \to S \) where \( I_{fg} = I_f \cap g(I_g) \), and \( f \Theta g \iff f(x) = g(x) \) for all \( x \) in some \( A \in \Sigma \). Let \( B_c = \{ f \in B | fJ \subseteq J^c \text{ for all left ideals } J \subseteq I_g \} \). Then \( B_c \) is a subsemigroup of \( B \) for if \( J \subseteq I_{fg}, f \subseteq \hat{f}(I_f \cap J^c) \subseteq (I_f \cap J^c)^c \subseteq S \cap \cap J^c = J^c \). (Note that this requires only that \( c \) is a closure operation.) Set \( Q_c(S) = B_c/\Theta_c \) where \( \Theta_c \) is the restriction of \( \Theta \) to \( B_c \). Then \( Q_c(S) \) is a subsemigroup of \( Q(S) \). Moreover, if \( c \) is bilateral, then for \( s \in S, I_s = S \) and \( As = A^c s = A^c \) so \( Q_c(S) \supseteq S \). We gather these results in the following

**Theorem:** If \( c \) is a closure operation on \( L \), then \( Q_c(S) \) is a subsemigroup of \( Q(S) \). If \( c \) is a bilateral closure operation on \( L \), then \( S \subseteq Q_c(S) \subseteq Q(S) \).

This construction enables one to prove the following results due to Murdoch [2] for rings.

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Proposition: For each bilateral closure $c$ on $L$, there is a unique bilateral closure $\overline{c}$ such that $Q_c(S) = Q_{\overline{c}}(S)$ and the mapping $c \mapsto \overline{c}$ is a closure operation on the lattice of bilateral closure operations on $L$. $\overline{c}$ is called a maximal bilateral closure operation on $L$.

Let $(M, \cap, \cup^*, \preceq)$ be the complete lattice of maximal bilateral closure operations on $L$ with $a \preceq b$ as before and $\cup^* c_\infty = \overline{\cup c_\infty}$. A subsemigroup $T$ of $Q(S)$ is a closure subsemigroup if $T = Q_c(S)$ for some (maximal) bilateral closure operation on $L$.

Proposition: The lattice $(M, \cap, \cup^*, \preceq)$ is anti-isomorphic to the lattice $C$ of closure subrings of $Q$ by the mapping $c \mapsto Q_c(S)$.

In order to construct $\overline{c}$ for a bilateral closure $c$, we let $J^Q = \bigcup_{f \in B_c} f(J \cap I_f)$. Then $q^*$ is a bilateral closure operation and $q^* = \overline{c}$ if $q^*$ is the closure operation given by

$\{J \in L | J^Q = J\}$.

Next let $R$ be any subset of $Q$ containing $S$ and construct

$J^Q = \bigcup_{a \in R} (J \cap I_{\infty})$ and so $q^* R^*$.

Theorem: Let $Q_c(R)$ be the minimal closure subsemigroup of $Q(S)$ containing $R$, then $Q_{q^* R}(S) = Q_c(S)$ and $q^* R = \overline{c}$.

3. Closure Operations and Green's Relation $L$. Let $Q_\Sigma(S)$ be the semigroup of quotients of $S$ with respect to $\Sigma$. Note that $S \subseteq Q_\Sigma(S)$ iff $\tau_S = \{(s,t) | \text{as = at for a in some } A \in \Sigma \}$ = id. Green's relation $L_S$ on $S$ is given by $L_S = \{(x,y) | Sx = Sy\}$. Define the relation $L_\Sigma$ on $S$ by $L_\Sigma = \{(x,y) | Ax = Ay \text{ for some } A \in \Sigma \}$. Note that $\tau_S \subseteq L_\Sigma$ but in general they
are unequal.

Recall [11] that $\Sigma$ has property (T) if every $Q_{\Sigma}(S)$ system is strongly torsion free and this is equivalent to $Q_{\Sigma}(S)i(A) = Q_{\Sigma}(S)$ for all $A \in \Sigma$ where $i(A)$ is the image of $A$ in $Q_{\Sigma}(S)$ under the canonical mapping of $S$ into $Q_{\Sigma}(S)$. Hence if $\Sigma$ has property (T) and $(s,t) \in L_{\Sigma}$, then $As = At$ so $i(A)i(s) = i(A)i(t)$ in $Q_{\Sigma}(S)$ so $Q_{\Sigma}(S)i(s) = Q_{\Sigma}(S)i(t)$ and $(i \times i)L_{\Sigma} \subseteq L_{Q}$. Moreover, since $S \in \Sigma$, $L_{S} \subseteq L_{\Sigma}$.

Proposition: If $\Sigma$ has property (T), then $(i \times i)L_{\Sigma} \subseteq L_{Q} \cap (i(s) \times i(s))$. In any case $L_{S} \subseteq L_{\Sigma}$.

Let $s \in S$, and suppose $S$ is strongly torsion free so that $S \subseteq Q_{\Sigma}(S)$. Then $[Q_{\Sigma}(S)s \cap S]^c = (Ss)^c$ for if $qs = ts \in S$, then there is $B \in \Sigma$ with $Bq \subseteq S$ so $Bqs \subseteq Ss$ so $qs \in (Ss)^c$. Now suppose $(s,t) \in L_{Q} \cap S \times S$, then $Q_{\Sigma}(S)s = Q_{\Sigma}(S)t$ so $(Ss)^c = (St)^c$.

Now suppose $\Sigma$ has property (T). Then if $(Ss)^c = (St)^c$, there are $A,B \in \Sigma$ with $As \subseteq St$ and $Bt \subseteq Ss$. Thus $Q_{\Sigma}(S)As \subseteq Q_{\Sigma}(S)St$ or $qs \subseteq Qt$ and $Qt \subseteq Qs$ so $(s,t) \in L_{Q} \cap S \times S$.

This same argument suffices to show:

Proposition: Let $S$ be strongly torsion free and $\Sigma$ have property (T), then

$$L_{Q} = \{ (\alpha, \beta) | (S\alpha^{-1})\alpha = (S\beta^{-1})\beta \}. $$

Finally let $\Sigma' = \langle M \rangle$ where $M$ is an idempotent two sided ideal of $S$. If $(\alpha, \beta) \in L_{Q}$, then $M\alpha \subseteq S\beta$ and by (Q3), $M\beta \subseteq M\alpha$. Since the argument is symmetric, $L_{Q} = \{ (\alpha, \beta) | M\alpha = M\beta \} = L_{\Sigma}$.

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