Michal Resl

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ON MODELS IN THE ALTERNATIVE SET THEORY

Michal RESL

Abstract: Connections between theories and set models are described. Some set structures keep continuous equivalence of indiscerninility. For every two metrics on a structure there exists a continuous motion of an observer.

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When approaching the theory of models in the light of the alternative set theory, we have a choice of several possibilities how to carry out our investigation. One of the ways is to follow the classical theory of models and to reformulate its results into the alternative set theory. This way proves to be technically successful though in the same time it may be the reason why it seems uninteresting. We are trying to adopt an alternative view towards mathematics as a whole and therefore the aim of the following article is to examine the new possibilities the alternative set theory introduces into the theory of models.

One of the new approaches is to differentiate the models on set (discreate) and proper semiset models. The theories not having set models (but having semiset ones) demand transcendence; these theories are "complex" ones. On the other hand, the theories having a set model can be considered as "simple" ones. Let us give one example; the theory of fields of characteristic zero has a set model (but not finite) whilst the theory of ordered fields of characteristic zero has no set model.

The structures can be enriched about the topological problematics. We can investigate which structures are compact, the continuous motion can also be studied.

This article touches only a narrow part of the problematics. The whole field of problems of the model theory in the alternative set theory leaves a vast field for further study.

1. In order we might talk about models we must at first define the notion of a language and a structure.

Let $\mathcal{L}$ be a given class of relation symbols, function symbols, and constant symbols. Assume that every relation symbol and function symbol of $\mathcal{L}$ has a natural number α associated to it; in such a case this relation or function symbol is said to be α-ary. Then $\mathcal{L}$ is called a language. Assume that the equation symbol is in every language and the empty set 0 is not a symbol in any language.

Similarly as in the classical model theory we can construct the class of formulas $L_{\mathcal{L}}$ of the language $\mathcal{L}$ (containing formulas of an arbitrary length - over N; this construction is made - for a certain language - in [2], the generalization to an arbitrary language is obvious). A formula is said to be finite if it has a finite length and contains only finite-ary relation and function symbols. The class of all finite formulas of a language $\mathcal{L}$ is denoted by $FL_{\mathcal{L}}$.

A class $\mathfrak{A}$ is called a structure in a language $\mathcal{L}$ if $\mathfrak{A}$ is a coding pair $\langle K,S\rangle$ so that: $K=\{0\}\cup\mathcal{L}$, $S''\{0\}\neq 0$ ($A=S''\{0\}$ is called the universe of $\mathfrak{A}$), if p is an α-ary relation symbol of $\mathcal{L}$ then $S''\{p\}$ is α-ary relation on A, if f is an α-ary function symbol of $\mathcal{L}$ then $S''\{f\}$ is an α-ary function on A, if c is a constant symbol of $\mathcal{L}$ then $S''\{c\}$ is a constant in A. For any symbol s of $\mathcal{L}$ we shall denote $s^{\mathfrak{A}}=S''\{s\}$ (the interpretation of s in $\mathfrak{A}$) for simplicity. Let us presuppose that the equation symbol is interpreted by the absolute equality. (This notion of a structure is more general as in [2].)

A structure $\mathfrak{A}$ in the language $\mathcal{L}$ is said to be a set structure if there exists a set expansion $\mathcal{L}'$ of $\mathcal{L}$ (i.e. $\mathcal{L}'$ is a set) and an expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to $\mathcal{L}'$ such that $\mathfrak{A}'$ is a set. For set structures we use letters $\mathfrak{M}, \mathfrak{N}$,... . The satisfaction relation $\models$ is defined (see [2]) between set structures and any formulas, and between any structures and finite formulas. The notion of elementary equivalence, elementary substructure and isomorphism is defined as in the classical model theory with respect to finite formulas.

A theory is any collection of finite formulas. A formula φ is provable in a theory T ($T\vdash\varphi$) if there exists a proof (of an arbitrary length) of φ in T. A finite formula φ is finitely provable in T ($T\vdash_F\varphi$) if there exists a proof of φ in T which has finite length and contains only finite formulas. A theory T is consistent (finitely consistent resp.) if not every (finite) formula of the language of T is (finitely) provable in T.

The two simple properties hold:

1) If $T \vdash \varphi$ and $\mathfrak{M} \models T$ then $\mathfrak{M} \models \varphi$.

2) If $T \vdash_F \varphi$ and $\mathfrak{A} \models T$ then $\mathfrak{A} \models \varphi$.

This means every theory having a set model is consistent and every theory having any model is finitely consistent. By the Gödel's theorem, every finitely consistent (at most countable) theory has (at most countable) model.

As there is no natural definition of the satisfaction relation between non-finite formulas and arbitrary structures, we have no semantical characterization of consistent theories (in contrast to finitely consistent theories). But in the alternative set theory finite formulas, finite proofs and finite consistency are in the centre of interest (and for these analogical results from the classical model theory hold), and non-finite formulas and proofs give us additional implements.

For an illustration consider the following example. Let α be an infinite natural number. Let $\mathcal{L}$ be the language containing constant symbols C_β for $\beta \leq \alpha$. Take $T = \{c_\beta = c_{\beta+1}; \beta < \alpha\} \cup \{c_0 \neq c_\alpha\}$. Then T is finitely consistent theory (it has a two point model) but not consistent because $T \vdash c_0 = c_\alpha$ & $c_0 \neq c_\alpha$. Thus T has no set model.

From now on every language will be at most countable with only finite-ary symbols. It follows that $\mathfrak{A}$ in $\mathcal{L}$ is a set structure iff $\mathfrak{A}$ has a set universe and the interpretation of any symbol of $\mathcal{L}$ in $\mathfrak{A}$ is a set.

<u>Theorem</u>. Every finitely consistent theory has a fully revealed model.

Proof: Take any model $\mathfrak{A}$ of the theory. Let $\mathfrak{A}^*$ be the revealment of $\mathfrak{A}$ (see [3]). Then $\mathfrak{A} \equiv \mathfrak{A}^*$.

<u>Theorem</u>. Every finitely consistent theory has a model with a set universe.

The proof follows from the preceding theorem and from the axiom of cardinalities.

From now to the end of this first section assume that every language contains no function symbols.

<u>Theorem</u>. A theory T has a set model iff every finite $T' \subseteq T$ has a set model.

The proof follows immediately from the prolongation axiom.

Lemma. Let $\mathfrak{A}$ be a countable structure. Then there exists a sequence of set structures $\{\mathfrak{M}_n; n \in FN\}$ such that $\mathfrak{M}_n \subseteq \mathfrak{M}_{n+1} \subseteq \mathfrak{A}$ for all n and $\mathfrak{A} = \bigcup \{\mathfrak{M}_n; n \in FN\}$.

Proof. Let $A = \{a_n; n \in FN\}$ be the universe of $\mathfrak{A}$. Take $M_n = \{a_0, \dots, a_n\}$ the universe of $\mathfrak{M}_n$.

<u>Theorem</u>. Let T be a finitely consistent theory having only Σ_2-axioms. Then there exists a set model of T.

Proof. It is sufficient to prove this theorem for a finite T and moreover, we may assume $T = \{\varphi\}$ where φ is a finite Σ_2-formula. Let $\mathfrak{A}$ be a countable model of T and $\mathfrak{A} = \bigcup \{\mathfrak{M}_n; n \in FN\}$. As $\mathfrak{A} \models \varphi$ and φ is Σ_2, there exists a finite number n such that $\mathfrak{M}_n \models \varphi$.

Let T be the theory of linear order without endpoints.

Obviously T has no set model and T has Π_2-axioms. This means we cannot get a stronger theorem.

Corollary. Let T have Σ_2-axioms only. If T is finitely consistent then T is consistent.

Corollary. Let $\mathfrak{A}$ be a countable structure. Then there exists a set structure $\mathfrak{M} \supseteq \mathfrak{A}$ such that for all Σ_2-formulas $\varphi(x,\cdot)$ and all $a,\cdot \in A$, if $\mathfrak{A} \models \varphi(a,\cdot)$ then $\mathfrak{M} \models \varphi(a,\cdot)$.

Theorem. Let T be a finitely consistent theory in a language containing only symbols for unary relations, equation and constants. Then T has a set model.

Proof. Assume T is finite. We can replace every constant symbol in T by one point unary relation symbol, so assume we have no constant symbol in our language $\mathcal{L}$. Let $R_1,\ldots$ $\ldots,R_m$ be a list of all unary relation symbols in $\mathcal{L}$. Add new unary relation symbols $R_{m+1},\ldots,R_{2m}$ and axioms $(\forall x)$ $(R_i(x) \equiv \neg R_{m+i}(x)$, $i=1,\ldots,m$. Let T' be this new theory in $\mathcal{L}'$. Then T' is finitely consistent. Set $w=\{u;u \subseteq \{1,\ldots$ $\ldots,2m\} \,\&\, |u|=m\}$. For $u \in w$ denote $\varphi_u(x) \equiv \bigwedge_{i\in u} R_i(x)$. Take $\mathfrak{A}$ a countable model of T' and $\mathfrak{B}$ a fully revealed model of $Th(\mathfrak{A})$. We may assume $\mathfrak{A} \preceq \mathfrak{B}$. If $\mathfrak{C}$ is any structure in $\mathcal{L}'$, let $C_u=\{a;a\in C \,\&\, \mathfrak{C} \models \varphi_u(a)\}$ for $u \in w$. As $\mathfrak{B}$ is fully revealed, there exists $\mathfrak{M}$ such that $\mathfrak{A} \subseteq \mathfrak{M} \subseteq \mathfrak{B}$ and $\{M_u:$ $:u\in w\}$ is a partition of M. For all $u \in w$ we have $M_u \approx B_u$ and we are able to construct an isomorphism F from $\mathfrak{M}$ onto $\mathfrak{B}$ such that for $u\in w$, $F''M_u=B_u$. It follows $\mathfrak{M} \models T$.

Remark. Assuming T is finite satisfying the assumptions of the preceding theorem there exists a finite model of T because the model $\mathfrak{M}$ constructed above can be arbitrary small infinite.

We know there are only a few theories having a set model. The question is how is it possible to approximate semiset models by set structures. We know every finitely consistent theory has at most countable model and every countable structure is a sum of a sequence of set structures. Let us try to approximate countable models by these set structures. The following theorem holds.

Theorem. Let $\mathfrak{A}$ be a countable structure, $\mathfrak{A} = \bigcup\{\mathfrak{M}_n; n \in FN\}$ be a sum of increasing structures. Let φ be a finite formula, $\varphi \equiv (\forall x_1)(\exists y_1)\ldots(\forall x_r)(\exists y_r)\psi(x_1,\ldots,x_r,y_1,\ldots \ldots,y_r)$ where ψ is an open formula (possibly with parameters from $\mathfrak{A}$). Then the following are equivalent

1) $\mathfrak{A} \models \varphi$

2) $(\forall n_1)(\exists k_1)\ldots(\forall n_r)(\exists k_r)\,\mathfrak{A} \models (\forall x_1 \in M_{n_1})(\exists y_1 \in M_{k_1})\ldots \ldots(\forall x_r \in M_{n_r})(\exists y_r \in M_{k_r})\psi$ (where $\mathfrak{A}$ in 2) can be replaced by $\mathfrak{M}_m$, $m=\max\{n_1,\ldots,n_r,k_1,\ldots,k_r\}$ or any $\mathfrak{M} \supseteq \mathfrak{A}$).

Proof. Assume at first M to be a finite set and χ to be a normal formula of AST. Then the two facts hold.

Fact 1. If $(k \leq k' \,\&\, \chi(x,k)) \longrightarrow \chi(x,k')$, then $(\forall x \in M)(\exists k)\chi(x,k) \equiv (\exists k)(\forall x \in M)\chi(x,k)$.

Proof. Assume $(\forall x \in M)(\exists k)\chi(x,k)$. Let $f: M \longrightarrow FN$ be a function such that $(\forall x \in M)\chi(x,f(x))$. Take $k=\max(f''M)$. Then $(\forall x \in M)\chi(x,k)$.

Fact 2. If $(n' \leq n \,\&\, \chi(y,n)) \longrightarrow \chi(y,n')$, then $(\exists y \in M)(\forall n)\chi(y,n) \equiv (\forall n)(\exists y \in M)\chi(y,n)$.

This fact follows immediately from the fact 1.

Let us turn to the proof of the theorem. We use an induction on r. The statement $\mathcal{A} \models \varphi$ is equivalent to

$$(\forall n_1)(\forall x_1 \in M_{n_1})(\exists k_1)(\exists y_1 \in M_{k_1})\ \mathcal{A} \models (\forall x_2)(\exists y_2)\dots(\forall x_r)(\exists y_r)\psi \ .$$

Using the fact 1 we get an equivalent statement

$$(\forall n_1)(\exists k_1)(\forall x_1 \in M_{n_1})(\exists y_1 \in M_{k_1})\ \mathcal{A} \models (\forall x_2)(\exists y_2)\dots(\forall x_r)(\exists y_r)\psi \ .$$

This statement is equivalent by the induction hypothesis to

$$(\forall n_1)(\exists k_1)(\forall x_1 \in M_{n_1})(\exists y_1 \in M_{k_1})(\forall n_2)(\exists k_2)\dots(\forall n_r)(\exists k_r)$$
$$\mathcal{A} \models (\forall x_2 \in M_{n_2})(\exists y_2 \in M_{k_2})\dots(\forall x_r \in M_{n_r})(\exists y_r \in M_{k_r})\psi \ .$$

By successive using the facts 1, 2 we are able to move the block $(\forall x_1 \in M_{n_1})(\exists y_1 \in M_{k_1})$ behind $\mathcal{A} \models$. We have got an equivalent statement

$$(\forall n_1)(\exists k_1)\dots(\forall n_r)(\exists k_r)\ \mathcal{A} \models (\forall x_1 \in M_{n_1})(\exists y_1 \in M_{k_1})\dots$$
$$\dots(\forall x_r \in M_{n_r})(\exists y_r \in M_{k_r})\psi \ .$$

Let us notice now what would happen if we took function symbols in our languages. Let $\mathcal{A}$ be a structure with the universe FN and the successor function. Then $\mathcal{A}$ has no set substructure, thus $\mathcal{A}$ cannot be expressed as a countable sum of a chain of set structures. From the similar reason the theorem about theories having only Σ_2-axioms would not hold (take the theory of linear order with the successor function).

2. Let us enrich our structures with topological properties. By a topology we mean in this article an equivalence relation which is a π-class. In this and next section let us restrict to structures with a set universe (every

finitely consistent theory has such a model). In contrast to the first section we admit function symbols in a language.

Let $\mathcal{A}$ be a structure in a language $\mathcal{L}$, $\sim$ a topology on A. The symbols $\bar{a}$, $\bar{b}$,... denote finite ordered sequences of points of A. If $\bar{a} = \langle a_1, \ldots, a_n \rangle$, $\bar{b} = \langle b_1, \ldots, b_n \rangle$ then $\bar{a} \sim \bar{b}$ means $a_1 \sim b_1 \,\&\, \ldots \,\&\, a_n \sim b_n$. A function F on A is continuous in $\sim$ if for all $\bar{a}$, $\bar{b}$, if $\bar{a} \sim \bar{b}$ then $F(\bar{a}) \sim F(b)$. $\mathcal{A}$ is said to be continuous in $\sim$ (or $\sim$ is continuous in $\mathcal{A}$) if the interpretations of all function symbols of $\mathcal{L}$ in $\mathcal{A}$ are continuous in $\sim$ and for any relation symbol R of $\mathcal{L}$ different from equation and all $\bar{a}$, $\bar{b}$, if $\bar{a} \sim \bar{b}$ then $\mathcal{A} \models R(\bar{a}) \equiv R(\bar{b})$.

<u>Theorem</u>. For every set structure $\mathfrak{M}$ there exists a coarsest topology continuous in $\mathfrak{M}$ and moreover, this topology is a countable intersection of set equivalences.

Proof. Let $\{R_n; n \in FN\}$, $\{F_n; n \in FN\}$ be a list of all relation and function symbols resp. of the language of $\mathfrak{M}$. For all n take $\theta_n = \{\langle a,b \rangle \in M^2; (\forall i \leq n)(\forall \bar{d}_1, \bar{d}_2 \in M)(\mathfrak{M} \models R_i(\bar{d}_1, a, \bar{d}_2) \equiv R_i(\bar{d}_1, b, \bar{d}_2))\}$ and define $a \overset{n\mathfrak{M}}{\sim} b$ iff $(\forall n)(\langle a,b \rangle \in \theta_n)$. Now define by induction $a \sim_0 b$ iff $\langle a,b \rangle \in M^2$, $a \sim_{n+1} b$ iff $\langle a,b \rangle \in \theta_n \,\&\, a \sim_n b \,\&\, (\forall i \leq n)(\forall \bar{d}_1, \bar{d}_2 \in M)(F_i^{\mathfrak{M}}(\bar{d}_1, a, \bar{d}_2) \sim_n \sim_n F_i^{\mathfrak{M}}(\bar{d}_1, b, \bar{d}_2))$ and define $a \overset{\mathfrak{M}}{\sim} b$ iff $(\forall n)(a \sim_n b)$. Then $\overset{\mathfrak{M}}{\sim}$ is the requested topology.

We know now that every set structure keeps its own continuous topology. The question is for which structures there exists a continuous equivalence of indiscernibility (i.e. compact topology).

Theorem. Let $\mathfrak{M}$ be a set structure in a language containing symbols for unary relations, unary functions, constants and equation only. Then there exists a totally disconnected equivalence of indiscernibility continuous in $\mathfrak{M}$.

Proof. We shall show that the topology $\overset{\mathfrak{M}}{\sim}$ described above is compact. First we show the topology $\overset{p\mathfrak{M}}{\sim}$ is compact. Every unary relation divides M into two parts, thus $\overset{p\mathfrak{M}_n}{\sim}$ is compact where $\mathfrak{M}_n$ is the reduct of $\mathfrak{M}$ to the language containing the n-th relation symbol only. The topology $\overset{p\mathfrak{M}}{\sim}$ is the intersection of $\overset{p\mathfrak{M}_n}{\sim}$'s and thus compact. Now we are able to show by induction that every $\sim_n$ is compact. Assume $\sim_n$ is compact and take $u \subseteq M$ infinite. There exists an infinite $u_1 \subseteq u$ such that for all $a,b \in u_1$ we have $a \overset{p\mathfrak{M}}{\sim} b$ and $a \sim_n b$. Let $\{v_\beta ; \beta \in \alpha\}$ be a partition of u_1 into monads by the set equivalence $\sim_{n+1}$. It is sufficient to show that for some $\beta \in \alpha$, v_β is infinite. Assume v_β is finite for all $\beta \in \alpha$. Let v be a selector of the partition $\{v_\beta , \beta \in \alpha\}$. As u_1 is infinite, the set v is infinite. For $a,b \in v$, $a \neq b$ we have $a \not\sim_{n+1} b$ and $(\exists i \leq n)(F_i^{\mathfrak{M}}(a) \not\sim_n F_i^{\mathfrak{M}}(b))$. For $a \in v$ define $g(a) = \langle F_0^{\mathfrak{M}}(a), \dots, F_n^{\mathfrak{M}}(a) \rangle$. Let $\langle M^{n+1}, \sim \rangle$ be the n+1-th power of the space $\langle M, \sim_n \rangle$, $\sim$ is the product topology. For $a,b \in v$, $a \neq b$ we have $g(a) \not\sim g(b)$ and thus the set $g''v$ is an infinite set in the compact topology $\sim$, a contradiction. Finally, the topology $\overset{\mathfrak{M}}{\sim}$ is an intersection of the topologies $\sim_n$ and so compact.

The following theorem shows that we are not able to prove anything more about the topology $\overset{\mathfrak{M}}{\sim}$.

Theorem. Let $\sim$ be a topology on a set M such that $\sim$

is a countable intersection of set equivalences. Then

1) There exists a set structure $\mathfrak{M}$ with the universe M such that $\sim$ is the coarsest topology continuous in $\mathfrak{M}$.

2) Moreover, if $\sim$ is compact then we are able to construct the structure in 1) with unary relations only.

Proof. Let $\sim$ be an intersection of set equivalences S_n, $n \in FN$.

1) Take the language with binary relation symbols P_n, $n \in FN$ and the structure $\mathfrak{M}$ such that $\mathfrak{M} \models P_n(a,b)$ iff $\langle a,b \rangle \in S_n$.

2) Assume $\sim$ is compact. Then any S_n is compact, too, and thus any equivalence S_n has only finite number of monads. Let $\{v_k; k \in FN\}$ be the list of all monads in all S_n's. Take the language with unary relation symbols P_n, $n \in FN$ and the structure $\mathfrak{M}$ so that $\mathfrak{M} \models P_n(a)$ iff $a \in v_n$.

3. Let us consider now the following situation. Imagine an observer observing some structure from some place. The observer can distinguish distances among points in the structure (described by a metric). Now imagine the observer is continuously moving. Then the distances among points are in a continuous motion.

<u>Definition</u>. 1) Let A be a set, ρ, ρ' two metrics on A. A set sequence $\{\rho_\alpha; \alpha \leq \vartheta\}$ is said to be a motion of an observer from ρ to ρ' on A in the time ϑ if $\rho_0 = \rho$, $\rho_\vartheta = \rho'$, $\rho_\alpha \doteq \rho_{\alpha+1}$ for $\alpha \in \vartheta$, where $\doteq$ is the Euclidean equivalence of indiscernibility and $\rho_\alpha \doteq \rho_{\alpha+1}$ means

$\rho_\alpha(a,b) \doteq \rho_{\alpha+1}(a,b)$ for all $a,b \in A$.

2) Let $\mathfrak{A}$ be a structure with a set universe A, ρ, ρ' two metrics continuous in $\mathfrak{A}$ (i.e. $\mathfrak{A}$ is continuous in the topologies generated by these metrics). A set sequence $\{\rho_\alpha ; \alpha \leq \vartheta\}$ is said to be a motion of an observer from ρ to ρ' on $\mathfrak{A}$ in the time ϑ if $\{\rho_\alpha ; \alpha \leq \vartheta\}$ is a motion of an observer from ρ to ρ' on A in the time ϑ and for all $\alpha \leq \vartheta$, $\mathfrak{A}$ is continuous in ρ_α.

Consider $\{\rho_\alpha ; \alpha \leq \vartheta\}$ is a motion of an observer on $\mathfrak{A}$. What is the connection between the structures $\mathfrak{A}/\rho_0$, $\mathfrak{A}/\rho_\vartheta$ (i.e. factorstructures w.r.t. the topologies generated by these metrics) ? One can prove that there is an infinite α such that $\rho_0 \doteq \rho_\alpha$. But we shall prove there need not be any connection between metrics ρ_0, ρ_ϑ.

Lemma. Let ρ be a metric on a set A. Then there exists a metric ρ^* on A and a motion of an observer $\{\rho_\alpha ; \alpha \leq \vartheta\}$ from ρ to ρ^* on A such that the three conditions hold:

1) $\rho^*(a,b) \leq 1$ for all $a,b \in A$
2) $\rho_{\alpha+1}(a,b) \leq \rho_\alpha(a,b)$ for $\alpha < \vartheta$, $a,b \in A$
3) all metrics ρ_α, $\alpha \leq \vartheta$ generate the same topology.

Proof. For $a,b \in A$ define $\rho^*(a,b) = \min\{1, \rho(a,b)\}$. ρ^* is a metric generating the same topology as ρ. The motion of an observer we construct by induction. Take $\rho_0 = \rho$. Take an infinite δ such that $\delta \geq |A|^2$ and assume ρ_α is constructed. Define for $a,b \in A$

$$\bar{\rho}_{\alpha+1}(a,b) = \begin{cases} \rho_\alpha(a,b) - \frac{1}{\delta} & \text{if } \rho_\alpha(a,b) - \frac{1}{\delta} \geq \rho^*(a,b) \\ \rho_\alpha(a,b) & \text{otherwise .} \end{cases}$$

Now define $\rho_{\alpha+1}(a,b)=$

$$=\min\{\Sigma_{\gamma\in\beta}\bar{\rho}_{\alpha+1}(a_\gamma,a_{\gamma+1});\ \beta\in N\ \&\ a_0=a\ \&\ a_\beta=b\ \&\ (\forall\gamma\leq\beta)(a_\gamma\in A)\}.$$

It is easy to see that $\rho_{\alpha+1}$ is a metric, $\rho_{\alpha+1}(a,b)\leq\leq\rho_\alpha(a,b)$ and $\rho_{\alpha+1}\doteq\rho_\alpha$. It is easy to prove by induction on α that $\rho^*(a,b)\doteq\rho_\alpha(a,b)$ for all a, b. Take $\eta\geq\geq\max\{\rho(a,b)-\rho^*(a,b);\ a,b\in A\}$, $\vartheta=\eta\cdot\delta+1$ and the construction of the ρ_α's stop at $\rho_{\eta\cdot\delta}$. We show $\rho^*\doteq\doteq\rho_{\eta\cdot\delta}$. If not, then for some $a,b\in A$, $\rho^*(a,b)+\frac{1}{\delta}<<\rho_{\eta\cdot\delta}(a,b)$, thus for all $\alpha<\eta\cdot\delta$, $\rho^*(a,b)\doteq\rho_\alpha(a,b)--\frac{1}{\delta}$, so $\rho^*(a,b)<\rho_{\eta\cdot\delta}(a,b)\doteq\rho(a,b)-\eta$, a contradiction with the choice of η. Define $\rho_\vartheta=\rho^*$.

<u>Theorem</u>. Let $\mathfrak{A}$ be a structure with a set universe, continuous in metrics ρ, ρ'. Then there exists a motion of an observer from ρ to ρ' on $\mathfrak{A}$. Moreover, if both ρ, ρ' are compact then every member of this motion is a compact metric.

Proof. By the preceding lemma we can suppose both ρ, $\rho'\leq 1$. We construct a motion of an observer from ρ to $\rho+\rho'$ as follows. Take δ infinite. Define $\rho_\alpha(a,b)==\rho(a,b)+\frac{\alpha}{\delta}\rho'(a,b)$, $a,b\in A$, $\alpha\leq\delta$. Denote $X=\{\alpha;\ \frac{\alpha}{\delta}\doteq 0\}$. For $\alpha\in X$, $\rho_\alpha(a,b)\doteq 0$ iff $\rho(a,b)\doteq 0$ and for $\alpha\notin X$, $\alpha\leq\delta$, $\rho_\alpha(a,b)\doteq 0$ iff $\rho(a,b)+\rho'(a,b)\doteq 0$. Thus for $\alpha\in X$, ρ_α generates the same topology as ρ does and for $\alpha\notin X$, ρ_α generates the same topology as $\rho+\rho'$ does (which is the intersection of topologies generated by ρ and ρ'). That is why $\mathfrak{A}$ is continuous in every ρ_α. Similarly we construct a motion of an observer from ρ' to $\rho+\rho'$. If ρ, ρ' are compact then $\wp + \wp'$ is compact and every member of the motion is compact.

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Matematicko-fyzikální fakulta

Universita Karlova

Malostranské nám. 25, Praha 1

Československo

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