Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 1, 131--143

Persistent URL: `http://dml.cz/dmlcz/105982`

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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
21, 1 (1980)

ON MEASURES OF NONCOMPACTNESS IN BANACH SPACES

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Abstract: The paper deals with a new axiomatics for measures of noncompactness which seems to be useful in applications. A fixed point theorem of Darbo's type and existence theorem for ordinary differential equations in Banach spaces are derived.

Key words: Measure of noncompactness, Darbo property, fixed point, ordinary differential equation in Banach space.

Classification: 47H09, 47H10, 34G20

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1. Introduction. In the last years there have appeared a lot of papers concerned with the notion of so-called measure of noncompactness. The most expository papers on this topic are e.g. [3],[11]. The notion of the measure of noncompactness was defined in many ways. At first, K. Kuratowski [10] has introduced for the family of all bounded subsets of metric space (M,ρ) the function $\alpha(X)$ defined below, which is a kind of a measure of noncompactness

$$\alpha(X) = \inf[\varepsilon > 0: X \text{ can be covered with a finite number of sets of diameter smaller than } \varepsilon] .$$

Another measure of noncompactness is so-called ball measure (or Hausdorff measure). It is defined by the formula

$$\chi(X) = \inf[\varepsilon > 0: X \text{ can be covered by a finite number of balls of radii smaller than } \varepsilon] .$$

This measure was introduced by Gohberg, Goldenštein, Markus [8], Sadovskiĭ [12] and Goebel [6].

There are some other definitions of measure of noncompactness. Some of the authors were trying to introduce this definition by an axiomatic way [9],[11]. At first it appeared in the paper of Sadovskiĭ [11], but his axiomatics seems to be too general. In this paper we present another axiomatic approach which is useful in applications.

Almost all known measures of noncompactness possess the property that they are equal to zero on the family of all relatively compact sets in a given space.
In our paper this property is omitted. It is very fruitful for applying such measures to the fixed point theory, because it gives a good characterization for the solutions of some functional equations [1]. In addition, our definition is appropriate for obtaining the formulas for the measure of noncompactness in the spaces in which convenient criteria of compactness do not exist [1].

2. Notations. Let $(E, \|\ \|)$ be a Banach space with the zero element θ. We denote

$\mathfrak{M}_E$ - the family of all bounded and nonempty subsets of E,

$\mathfrak{N}_E$ - the family of all relatively compact and nonempty subsets of E,

$\mathfrak{M}_E^c$, $\mathfrak{N}_E^c$ - subfamilies of $\mathfrak{M}_E$, $\mathfrak{N}_E$ respectively, consisting of closed sets.

Let D denote the Hausdorff pseudometric on the family $\mathfrak{M}_E$. It is well known that D is a complete metric on $\mathfrak{M}_E^c$ and the metric space $(\mathfrak{N}_E^c, D)$ is a closed subspace of the space $(\mathfrak{M}_E^c, D)$. The closure of a set X, its diameter and its convex closure will be denoted, respectively, by $\overline{X}$, diam X, Conv X. If $X, Y \in \mathfrak{M}_E$, $\lambda, \mu \in R$, then

$$\lambda X + \mu Y = [\lambda x + \mu y : x \in X, y \in Y].$$

The closed ball with the center in x and of radius r will be denoted by $K(x,r)$. "The ball" centered at an arbitrary set X of radius r will be denoted by $K(X,r)$, i.e.

$$K(X,r) = \bigcup_{x \in X} K(x,r).$$

3. Measure of noncompactness. Axiomatic approach. Our axiomatics of measure of noncompactness consists of two parts.

Definition 1. We call the kernel of a measure of noncompactness any nonempty family $\mathcal{P} \subset \mathfrak{N}_E$ satisfying the following conditions:

1° $X \in \mathcal{P} \Rightarrow \overline{X} \in \mathcal{P}$,

2° $X \in \mathcal{P}$, $Y \subset X$, $Y \neq \emptyset \Rightarrow Y \in \mathcal{P}$,

3° $X, Y \in \mathcal{P} \Rightarrow \lambda X + (1-\lambda) Y \in \mathcal{P}$ for $\lambda \in \langle 0,1 \rangle$,

4° $X \in \mathcal{P} \Rightarrow$ Conv $X \in \mathcal{P}$,

5° $\mathcal{P}^c$ (i.e. collection of all compacts belonging to $\mathcal{P}$) is closed in $\mathfrak{M}_E^c$ with respect to the Hausdorff topology.

Definition 2. A function $\mu: \mathfrak{M}_E \to \langle 0, +\infty)$ is said to be a measure of noncompactness with kernel $\mathcal{P}$ (ker $\mu = \mathcal{P}$)

provided it satisfies the following conditions:

1° $\mu(X) = 0 \Longleftrightarrow X \in \mathcal{P}$,

2° $\mu(\overline{X}) = \mu(X)$,

3° $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,

4° $\mu(\mathrm{Conv}\, X) = \mu(X)$,

5° $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in \langle 0,1\rangle$,

6° if $X_n \in \mathfrak{M}_E^c$, $X_{n+1} \subset X_n$ for $n = 1,2,\ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

A measure such that for any $X \in \mathfrak{M}_E$ and $\lambda \in R$

7° $\mu(\lambda X) = |\lambda|\,\mu(X)$

is said to be homogeneous, and if it satisfies

8° $\mu(X + Y) \leq \mu(X) + \mu(Y)$

it is called subadditive. It is sublinear if both conditions $7^\circ, 8^\circ$ hold.

Notice that the Kuratowski's measure α and ball measure χ are sublinear measures of noncompactness with kernel $\mathfrak{N}_E$ (see e.g. [6],[11]). The simplest example of a measure with $\mathcal{P} \neq \mathfrak{N}_E$ is the diameter, diam X. Its kernel is the family of all one-point sets. Another example of such measures may be found in [1].

Observe now that each kernel $\mathcal{P}$ admits at least one measure.

<u>Theorem 1</u>. For any kernel $\mathcal{P}$ the function

$$\mu(X) = D(X,\mathcal{P}) = \inf[D(X,Y) : Y \in \mathcal{P}]$$

is a measure of noncompactness with kernel $\mathcal{P}$.

We omit the proof which is based on some properties of the function D ([1],[6]).

Now we prove a few lemmas describing some properties of measures of noncompactness. We assume that μ is an arbitrary measure with kernel $\mathcal{P}$.

Lemma 1. (cf. [2].) If $\varepsilon \in \langle 0,1)$ then

$$\mu(K(X,\varepsilon)) \le \mu(X) + \varepsilon\, \mu(K(X,1)).$$

Proof. It is easy to verify that the function

$$\varphi(t) = \mu(K(X,t)), \quad t \ge 0$$

is nondecreasing, convex and nonnegative, also continuous. Then

$$\frac{\mu(K(X,\varepsilon)) - \mu(X)}{\varepsilon} \le \mu(K(X,1)) - \mu(X) \le \mu(K(X,1)),$$

and the proof of our lemma is complete.

Lemma 2. If $\|X\| = \sup[\|x\| : x \in X] < 1$ then

$$\mu(X + Y) \le \mu(Y) + \|X\|\, \mu(K(Y,1)).$$

The proof of the above lemma is similar to the proof of Lemma 1.

Lemma 3. If $\{\theta\} \in \mathcal{P}$ then $\mu(tX) \le t\,\mu(X)$ for $t \in \langle 0,1\rangle$.

Indeed we have

$$\mu(tX) = \mu((1-t)\{\theta\} + tX) \le (1-t)\mu(\{\theta\}) + t\,\mu(X) = t\,\mu(X).$$

Lemma 4. Let $t_1, t_2, \ldots, t_n$ be given nonnegative reals such that $\sum_{i=1}^{n} t_1 \le 1$ and let $\{\theta\} \in \mathcal{P}$. Then

$$\mu\Big(\sum_{i=1}^{n} t_i X_i\Big) \le \sum_{i=1}^{n} t_i\, \mu(X_i).$$

Proof. If $\sum_{i=1}^{n} t_i = 0$ then the inequality is obvious. Let $\sum_{i=1}^{n} t_i > 0$. Denoting $\lambda_i = \frac{t_i}{\sum_{k=1}^{n} t_k}$, we have with respect to Lemma 3 and the axiom 5° (Definition 2):

$$\mu\Big(\sum_{i=1}^{n} t_i X_i\Big) = \mu\Big(\sum_{i=1}^{n} t_i\Big)(\lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n)) \le$$

$$\leq \sum_{i=1}^{n} t_i \, [\lambda_1 \mu(X_1) + \lambda_2 \mu(X_2) + \ldots + \lambda_n \mu(X_n)] = \sum_{i=1}^{n} t_i \, \mu(X_i).$$

4. Operators satisfying Darbo condition and a fixed point theorem. Let E_1, E_2 be Banach spaces and let μ_1, μ_2 be some measures of noncompactness in E_1, E_2 respectively. We will consider the operators defined on a subset F of E_1 with values in E_2. In the next we assume that those operators are continuous.

Definition 3 (see e.g.[2]). We say that the operator $T:F \longrightarrow E_2$ satisfies the Darbo condition with a constant k with respect to the measures μ_1, μ_2 if for any set $X \subset F$ such that $X \in \mathfrak{M}_{E_1}$, its image $TX \in \mathfrak{M}_{E_2}$ and

$$\mu_2(TX) \leq k \, \mu_1(X).$$

If $k < 1$ then we call T a μ_1-μ_2-contraction or shortly, μ-contraction if $E_1 = E_2$ and $\mu_1 = \mu_2 = \mu$.

Notice that $T:F \subset E \longrightarrow E$ is a contraction with respect to the diameter if and only if T is a contraction in the classical sense.

We prove now a fixed point theorem of Darbo type (cf. [5],[1]).

Theorem 2. Let $C \in \mathfrak{M}_E$, Conv $C = C$ and let $T:C \longrightarrow C$ be a μ-contraction, where μ is an arbitrary measure of noncompactness. Then T has at least one fixed point which belongs to $\ker \mu$.

Proof. Consider the sequence of sets $C_0 = C$, $C_{n+1} = \text{Conv}\, TC_n$. Then

$$\mu(C_{n+1}) = \mu(\text{Conv}\, TC_n) = \mu(TC_n) \leq k \, \mu(C_n).$$

Hence

$$\mu(C_n) \leq k^n \mu(C_0)$$

and consequently

$$\lim_{n \to \infty} \mu(C_n) = 0.$$

Because $C_{n+1} \subset C_n$ and $T: C_n \longrightarrow C_n$ for all $n = 0,1,2,\ldots$, then $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is a convex closed set belonging to $\ker \mu$ and invariant under T. The classical Schauder fixed point theorem completes the proof.

5. Some properties of operators satisfying Darbo condition. Let, as earlier, E_1, E_2 denote Banach spaces with given measures of noncompactness μ_1, μ_2, respectively. We give some properties of the operators satisfying the Darbo condition.

Theorem 3. If $T_1, T_2 : F \subset E_1 \longrightarrow E_2$ satisfy the Darbo condition with a constant k, then the operator

$$T_\lambda = \lambda T_1 + (1-\lambda)T_2, \text{ for } \lambda \in \langle 0,1 \rangle$$

satisfies the Darbo condition with the same constant k.

The proof is obvious.

Theorem 4. Let $\{T_n\}$ be a sequence of operators defined on $F \subset E_1$ and taking values in E_2, which satisfy the Darbo condition with the same constant k. If T_n converges uniformly on any bounded subset of F to an operator T, then T satisfies the Darbo condition with constant k.

Proof. Let $\varepsilon \in (0,1)$. Then for any bounded set $X \subset F$ there exists an integer n_0 such that for any $n \geq n_0$

$$\sup [\| T_n x - Tx \| : x \in X] \leq \varepsilon .$$

Hence

$$TX \subset K(T_n X, \varepsilon).$$

Thus in view of Lemma 1 we obtain

$$\mu_2(TX) \le \mu_2(K(T_n X,\varepsilon)) \le \mu_2(T_n X) + \varepsilon\mu_2(K(T_n X,1)) \le$$
$$\le k\,\mu_1(X) + \varepsilon\mu_2(K(T_n X,1)),$$

for any $n \ge n_0$. Because the sequence $\{\mu_2(K(T_n X,1))\}$ is bounded and the above inequalities hold for arbitrary $\varepsilon \in (0,1)$, we finally have

$$\mu_2(TX) \le k\,\mu_1(X).$$

This ends the proof.

The above theorem was first proved by Daneš [4] for the case of the ball measure χ.

If we denote by $\mathfrak{X}$ the family of all bounded and continuous operators acting from $F \subset E_1$ into E_2 and by $\mathfrak{X}_k$ $(k \ge 0)$ its subfamily consisting of all operators satisfying the Darbo condition with constant k (with respect to the measures μ_1, μ_2), then in view of Theorems 3 and 4 we obtain that the family $\mathfrak{X}_k$ forms a convex and closed subset of the family $\mathfrak{X}$ (with respect to the topology of uniform convergence on bounded sets).

6. <u>An existence theorem for ordinary differential equation in Banach space.</u> In this section we shall give some applications of measures of noncompactness to the existence problem for ordinary differential equation. Our result extends a result of the works [2],[7].

Denote by $C = C(\langle 0,T\rangle, E)$ the space of all continuous functions defined on the interval $\langle 0,T\rangle$ with values in the Banach space E. For $x = \{x(t)\} \in C$ we define the usual maximum norm

$$\|x\| = \max [\|x(t)\|_E : t \in \langle 0,T\rangle].$$

For arbitrary $X \in \mathfrak{M}_C$ and $\varepsilon > 0$ we put

$$\omega(X,\varepsilon) = \sup_{x \in X} \{\sup [\|x(t) - x(s)\|_E : t,s \in \langle 0,T\rangle, |t-s| \leq \varepsilon]\}$$

and

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X,\varepsilon).$$

According to the Arzela theorem and result of [6] we can easily deduce that if E is a finite dimensional space then $\omega_0(X)$ is the measure of noncompactness in the space C and $\omega_0(X) = 2\chi(X)$. If E is an infinite dimensional space we must add a component which measures the noncompactness of cross-sections $X(t)$, where $X(t) = [x(t) : x \in X]$. Therefore, let μ_E be an arbitrary measure of noncompactness in E with kernel $\mathcal{P}_E$. We put

$$M(X) = \sup [\mu_E(X(t)) : t \in \langle 0,T\rangle] .$$

Finally let us define

$$\mu(X) = \omega_0(X) + M(X).$$

This function is a measure of noncompactness in the space C with kernel $\mathcal{P}_C$ consisting of all equicontinuous sets X such that $X(t) \in \mathcal{P}_E$ for any $t \in \langle 0,T\rangle$ [2].

It is worth to mention that the function $M(X)$ is the measure of noncompactness on the family $\mathfrak{M}_C^{eq}$ of equicontinuous sets (i.e. it satisfies the axioms of measure of noncompactness on this family).

Now we prove some generalization of Goebel-Rzymowski lemma [7]. First we denote

$$\int_0^t X(s)ds = [\int_0^t x(s)ds : x \in X].$$

Lemma 5. If $X \in \mathfrak{M}_C^{eq}$ and $\{\Theta\} \in \mathcal{P}_E$ then for any $t \in \langle 0, \min\{1,T\}\rangle$ the following inequality holds:

$$\mu(\int_0^t X(s)ds) \leq \int_0^t \mu(X(s))ds.$$

Proof. Taking an arbitrary $\varepsilon \in (0,1)$, in view of equicontinuity, we can choose points $0 = t_0 \leq \xi_1 \leq t_1 \leq \xi_2 \leq \leq \ldots \leq \xi_n \leq t_n = t$ so densely in $\langle 0,t\rangle$ that for all $x \in X$

$$\| \int_0^t x(s)ds - \sum_{i=1}^{n} x(\xi_i)(t_i - t_{i-1}) \| \leq \varepsilon .$$

Thus we get

$$\int_0^t X(s)ds) \subset [\int_0^t x(s)ds - \sum_{i=1}^{n} x(\xi_i)(t_i - t_{i-1}) : x \in X] +$$
$$+ [\sum_{i=1}^{n} x(\xi_i)(t_i - t_{i-1}) : x \in X] = A + B.$$

Now in view of Lemma 2 we obtain

$$\mu(A + B) \leq \mu(B) + \| A \| \mu(K(B,1)) \leq$$
$$\leq \varepsilon \mu(K(S,1)) + \mu([\sum_{i=1}^{n} x(\xi_i)(t_1 - t_{i-1}) : x \in X]).$$

Hence, by Lemma 4, we have

$$\mu(\int_0^t X(s)ds) \leq \sum_{i=1}^{n} (t_i - t_{i-1}) \mu(X(\xi_i)) + \varepsilon \mu(K(B,1)).$$

Densifying the partition of $\langle 0,t\rangle$ completes the proof.

Let us consider the ordinary differential equation

(1) $$x' = f(t,x)$$

with the Cauchy initial value condition

(2) $$x(0) = \Theta.$$

We shall assume that f is defined on $\langle 0,T\rangle \times E$, continuous and bounded.

Theorem 5. Let f be a uniformly continuous function on $\langle 0,T\rangle \times K(\Theta,r)$. Let $\|f(t,x)\| \le A$, $AT \le r$, $T \le 1$ and

$$\mu(f(t,X)) \le p(t)\,\mu(X)$$

for any $X \in \mathfrak{M}_C$ and for almost all $t \in \langle 0,T\rangle$, where p(t) is a Lebesgue integrable function on $\langle 0,T\rangle$. Then the equation (1) has at least one solution x satisfying the condition (2) and such that $x(t) \in \mathcal{P}_E$ for $t \in \langle 0,T\rangle$.

Proof. Let $X_0 \subset C(\langle 0,T\rangle,E)$ be the set of all functions x such that $x(0) = \Theta$ and $\|x(t) - x(s)\|_E \le A|t - s|$. X_0 is closed, bounded, convex and equicontinuous. The transformation

$$(Fx)(t) = \int_0^t f(s,x(s))ds$$

maps continuously X_0 into itself. Thus our problem is equivalent to the existence of a fixed point of F.

Now, for any $X \in \mathfrak{M}_C^{eq}$ and $\varkappa \ge 0$ put

$$\mu_\varkappa(X) = \sup\,[\mu(X(t))\,\exp(-\varkappa\int_0^t p(s)ds) : t \in \langle 0,T\rangle]\,.$$

We can easily check that $\mu_\varkappa(X)$ satisfies the axiom of measure of noncompactness on the family $\mathfrak{M}_C^{eq}$. Hence and with respect to Lemma 5, we obtain for any $X \in \mathfrak{M}_C^{eq}$

$$\mu((FX)(t)) = \mu(\int_0^t f(s,X(s))ds) \le \int_0^t \mu(f(s,X(s)))ds \le$$

$$\le \int_0^t p(s)\,\mu(X(s))ds \le \mu_\varkappa(X)\int_0^t p(s)\exp(\varkappa\int_0^t p(\tau)d\tau)ds \le$$

$$\le \exp(\varkappa\int_0^t p(s)ds)\,\frac{1}{\varkappa}\,\mu_\varkappa(X).$$

Dividing both sides by $\exp(\varkappa\int_0^t p(s)ds)$ and taking supremum on the left hand we obtain

$$\mu_\varkappa(FX) \le \frac{1}{\varkappa}\,\mu_\varkappa(X).$$

If $\varkappa > 1$ then F is a $\mu_\varkappa$-contraction and in view of Theorem 2, it has a fixed point x such that $x(t) \in \mathcal{P}_E$ for $t \in$ $\in \langle 0,T\rangle$. Thus the proof is complete.

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(Oblatum 31.5.1979)