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Counterexample to the regularity of weak solution of elliptic systems

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Abstract: In the paper there will be given an example of nonlinear elliptic system

\[ \sum_{i=1}^{m} D_i (a_i (\text{grad } u)) = 0, \ r = 1, \ldots, m \text{ on } \Omega = \{ x \in \mathbb{R}_n, |x| < 1 \}, \ u = u_0 \text{ on } \partial \Omega, \]

having analytic coefficients and unique solution with discontinuous but bounded first derivatives even in dimensions \( n = 3, 4 \). (For \( n = 5 \) an example of considered type was constructed by J. Nečas (see [10]).)

In the introduction we give a brief survey of the problem of regularity and counterexamples. In Chapter 1 there will be studied the counterexample mentioned above. In Chapter 2 we add some calculations omitted in Chapter 1 in details.

Key words: Regularity, elliptic systems

Classification: 35J60, 35D10

Introduction. The problem of regularity (or analyticity) of weak solutions of nonlinear elliptic systems can be traced to the beginning of this century - to the 19. D. Hilbert's problem and can be expressed by the question: Supposing \( a_i \) and \( u_0 \) in (1) to be analytic, is the weak solution \( u \) also analytic function? The history of this problem is described in several books and papers (see [5],[6],
[41], hence we will mention here only some crucial points. Very soon - in 1939 - the problem was solved positively for systems of equations of second order in plane by Ch.B. Morrey. Very important further step was made by E.De Giorgi and J. Nash in 1957. They proved regularity of solution of one equation of second order in the space $\mathbb{R}_n$ of arbitrarily high dimension $n$. Another positive result was proved by one of the authors (J. Nečas - in 1967) for equations of arbitrarily high order in plane. Almost immediately there appeared counterexamples (E.De Giorgi - 1968, E. Giusti, M. Miranda - 1968), showing that the situation of one equation of second order in $\mathbb{R}_n$ or of systems of arbitrary order in plane is in some sense exceptional and that there exist systems with analytic coefficients whose solutions are not even continuous ($x$). Unfortunately, these counterexamples have some disadvantages:

(i) They have analytic coefficients, they are naturally defined on Sobolev spaces $W_2^1$, but the corresponding operators are not differentiable on this space.

(ii) For low dimensions (which play the most important role in physics) it is unclear, whether the irregular solution is unique or if, perhaps, there could exist another regular solution of system in question ($xx$).

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($xx$) i.e. the typical quasilinear system

$$\sum_{i,j=1}^n \partial_{ij}(A_{ij}(u) \partial_j u_r) = 0 \text{ for } r = 1, \ldots, n.$$
In 1977 J. Nečas constructed a counterexample without these disadvantages and working in all dimensions $n \geq 5$, but the problem in $n = 3, 4$ still remained unsolved. The aim of this paper is to give a counterexample with unique irregular solution in $\mathbb{R}_n$ with dimensions $n \geq 3$.

Chapter 1

1.1. Notation. Let $\Omega = \{x \in \mathbb{R}^n; |x| < 1\}, u: \Omega \rightarrow \mathbb{R}^n$, $u = \{u_{ij}^n, i,j = 1\}$.

Let us denote

$$D^k u_{ij} = \frac{\partial u_{ij}}{\partial x_k}, \sigma_{ij}^k \text{ the Kronecker symbol},$$

$$\nabla_j u = \sum_{i=1}^{n^2} D^i u_{ji}, \Vert \nabla u \Vert^2 = \sum_{i=1}^{n^2} (\nabla_j u)^2, (\nabla u, \nabla v) = \sum_{i,j=1}^{n^2} \nabla_j u \nabla_j v,$$

for a fixed real number $\gamma$ let

$$\nabla_{ijk} u = D^k u_{ij} + \gamma (\sigma_{ij}^k \nabla_k u + \sigma_{ik}^j \nabla_j u + \sigma_{jk}^i \nabla_i u),$$

$$\Vert \sigma u \Vert^2 = \sum_{i,j,k=1}^{n^2} (\nabla_{ijk} u)^2, (\sigma u, \sigma v) = \sum_{i,j,k=1}^{n^2} \nabla_{ijk} u \nabla_{ijk} v.$$

1.2. System and its solution. Let $\gamma, \lambda, \nu$ be real numbers. We shall consider the system

$$\sum_{k=1}^{n^2} D^k \cdot D^k u_{ij} + \gamma (\sigma_{ij}^k \nabla_k u + \sigma_{ik}^j \nabla_j u + \sigma_{jk}^i \nabla_i u) +$$

$$+ \lambda \nabla_j u \nabla_j u \nabla_j u [1 + \Vert \nabla u \Vert^2]^{-1} +$$

$$+ \sigma_{ik}^j \nabla_j u \{\gamma(4+3 \gamma(n+2)) + 3 \lambda \nabla u \Vert^2[1 + \Vert \nabla u \Vert^2]^{-1} +$$

$$+ \nu \Vert \nabla u \Vert^4[1 + \Vert \nabla u \Vert^2]^{-2}\} = 0.$$

We shall prove that the function $u = \{u_{ij}^n, i,j = 1\}$...
(3) \( u_{ij}(x) = x_i x_j \) \(|x|^{-1} - \frac{1}{n} |x| \delta_{ij} \)

is a weak solution of the Dirichlet boundary problem for the system (2), i.e. for every infinitely differentiable function \( \varphi \) with compact support in \( \Omega \) the equality

\[
\left< Au, \varphi \right> = \int_{\Omega} \left\{ \nabla_i \nabla_j u + \lambda \nabla_i u \nabla_j u \nabla_k u \left[ 1 + \| \nabla u \|^2 \right]^{-1} \nabla_i \nabla_j u \nabla_k \varphi + v \| \nabla u \| \left[ 1 + \| \nabla u \|^2 \right]^{-2} (\nabla u, \nabla \varphi) \right\} = 0
\]

holds, if the numbers \( \lambda, \nu, \gamma \) satisfy the following conditions

(5) \( \lambda = [1 + (n - \frac{1}{2})^2] (n - \frac{1}{n} - \gamma) \),

(6) \( \nu = - (n - \frac{1}{n})^5 \gamma^2 (n + 1) (n - \frac{1}{n}) + \gamma (n^2 + 3n + 2) + 1 + \frac{1}{n} \times [1 + (n - \frac{1}{n})^2]^2 \).

1.3. Unicity of the solution. Unicity of the solution is an immediate consequence of the following inequality

(7) \( \langle DA(u) \varphi, \varphi \rangle \geq C \| \varphi \|^2 \left[ \frac{1}{2} \right]^{n^2} \),

holding for a positive constant \( C \) and a class of test functions \( \varphi \), and which implies that the operator \( A \) is strongly monotone. In fact, as it is proved in 2.3, we establish an algebraic condition of monotonicity, i.e. the integrand of \( \langle DA(u) \varphi, \varphi \rangle \) is greater than

\[
(1 - \frac{3}{4} \frac{A^2}{\nu}) \| \nabla \varphi \|^2.
\]

We have

\[
\langle DA(u) \varphi, \varphi \rangle = \int_{\Omega} \left\{ \| \nabla \varphi \|^2 + \lambda \left\{ \nabla_i u \nabla_j u \nabla_k \varphi + \nabla_i u \nabla_j \varphi \nabla_k u + \nabla_i \varphi \nabla_j u \nabla_k u \right\} \left[ 1 + \| \nabla u \|^2 \right]^{-1} \nabla_i \nabla_j u \nabla_k u \left( -2 (\nabla u, \nabla \varphi) \right) \right\} - 148 -
\]
Estimating the second term by Hölder inequality we get

\[
\langle DA(u) \varphi, \varphi \rangle \geq \int_\Omega (1 - \frac{3 \lambda^2}{4}) \| \partial^i \varphi \|^2.
\]

Taking in consideration the symmetry of the solution and of the system in the indices \( i, j \), we can choose the test function \( \varphi \) symmetric in \( i, j \), too. Thus it suffices to consider the test functions \( \varphi \) with only \( m = \frac{1}{2} n (n + 1) \) different components. For such functions \( \varphi \) we get (supposing that \( \gamma < 0 \))

\[
\| \partial^i \varphi \| \| \varphi \|_2^2 \geq (1 - 4 + \frac{n \gamma}{3(n + 2)}) \| \varphi \|_2^2 \| \varphi \|_2^2.
\]

Summarizing the inequalities (8) and (9) we obtain (7) with a constant \( C \) which is positive if

\[
1 - \frac{3 \lambda^2}{4} > 0,
\]

which implies the inequality

\[
\gamma < \frac{-2}{n + 3};
\]

and if

\[
1 - \frac{3 \lambda^2}{4} > 0.
\]

The second condition implies that \( \gamma \in (\gamma_1, \gamma_2) \) where (for \( n = 3 \))

\[
\gamma_1 = \frac{-27 \pm 2 \sqrt{42}}{102}
\]

is approximately \( \gamma_1 = -0.39, \gamma_2 = -0.13 \). Analogous numerical results show that the counterexample works in dimen-

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sions $m = 4,5$. For higher dimensions the function
\[ \gamma_1(m) = \frac{2}{m+3} \]
is a decreasing function of variable $n$. It proves that for all $m \geq 3$ we can choose $\gamma$ so that the function $u$ given by (3) is the unique solution of the system (2) with analytic coefficients and linear growth. Moreover, $u$ is the solution of the Dirichlet boundary problem with analytic boundary condition $u_0$.

Chapter 2

2.1. Deduction of $\lambda(\gamma)$ and $\nu(\gamma)$. The system (4) can be written in the form
\[ \int \{ \hat{\phi}_{ijk} v_{ijk} - \psi_i v_i \} dx = 0, \quad \phi_{ij} = \phi_{ji} \in \mathcal{D}. \]
By means of the Gauss formula we deduce from (10) the system in a strong form
\[ \begin{aligned}
\sum_{i,j} \Phi_{ijk} + \nu B^j (\hat{\phi}_{kii} + \hat{\phi}_{kik} + \hat{\phi}_{ikk}) + \nu^j \psi_i &= 0, \\
(i,j &= 1,\ldots,n),
\end{aligned} \]
remembering that
\[ \begin{aligned}
\phi_{ijk} &= v_{ijk} u + \lambda [1 + \|v\|^2]^{-1} v_i u v_j u v_k u, \\
\psi_i &= \nu [1 + \|v\|^2]^{-2} \|v\|^4 v_i u.
\end{aligned} \]
We want to choose the parameters $\nu, \lambda, \gamma$ in such a way that the function
\[ u_{ij}(x) = \frac{x_i x_j}{|x|} - \frac{1}{n} \delta_{ij} |x|, \quad (i,j = 1,\ldots,n) \]
would be the solution of (11).
After tedious but not difficult calculations we get the following expressions for $\Phi$ and $\Psi$ (see (12)) in case of function $u$ given by (13):

$$
\Phi_{ijk} = |x|^{-2}[a(\sigma_{ik}x_j + \sigma_{jk}x_i) + b \sigma_{ij}x_k] + c \frac{x_i x_j x_k}{|x|^3},
$$

$$
\Psi_i = (n - \frac{1}{n})^5[1 + (n - \frac{1}{n})^2]^{-2} \frac{x_i}{|x|},
$$

where

$$
a = 1 + \gamma(n - \frac{1}{n}), \quad b = - \frac{1}{n} + \gamma(n - \frac{1}{n}),
$$

$$
c = \lambda[1 + (n - \frac{1}{n})^2]^{-1}(n - \frac{1}{n})^3 - 1.
$$

Substituting (14) and (15) into (11) and differentiating we put the coefficients of $\sigma_{ij} x^{-1}$ and $x_i x_j x^{-3}$ equal zero. Thus we obtain

$$
2a + (n - 1)b + \gamma[2(2 + n)a + (2 + n)b + 3c] = 0,
$$

$$
-2a + (n - 1)c - \gamma[2(2 + n)a + (2 + n)b + 3c] = 0.
$$

From here it follows immediately that

$$
\lambda = [1 + (n - \frac{1}{n})^2] (n - \frac{1}{n})^{-2} \left(\frac{1}{n-1} - \gamma\right),
$$

$$
\gamma = -(n - \frac{1}{n})^{-5} \{3 \gamma^2 (n+1)(n - \frac{1}{n}) + \gamma(n^2 + 3n + 2) + (1 + \frac{1}{n})^2 \} \times [1 + (n - \frac{1}{n})^2]^2.
$$

2.2. Equivalent norms. We are to formulate sufficient conditions on parameter $\gamma$ under which there exists a constant $c_\gamma > 0$ such that

$$
\|Du\|^2 \geq c_\gamma \|Du\|^2.
$$
where \( \|Du\|^2 = \sum_{i,j,k} (D^k u_{ij})^2 \), \( \|\delta u\|^2 = \sum_{i,j,k} (\nabla_{ijk} u)^2 \).

It is
\[
\|\delta u\|^2 = \sum_{i,j,k} \left[ D^k u_{ij} + \gamma (\delta_{ij} \nabla_k u + \delta_{ik} \nabla_j u + \delta_{jk} \nabla_i u) \right]^2 \geq \]
(supposing that \( \gamma < 0 \)) \( \|Du\|^2 + 2\gamma \sum_{i,j,k} |D^k u_{ij}| |\nabla_k u| + [4\gamma + 3\gamma^2(n + 2)] \|\nabla u\|^2 \geq \|Du\|^2 + 2\gamma \sqrt{n} \|Du\| \|\nabla u\| + [4\gamma + 3\gamma^2(n + 2)] \|\nabla u\|^2 = [4\gamma + 3\gamma^2(n + 2)] \|\nabla u\|^2 + (\gamma \sqrt{n} \|Du\|)[4\gamma + 3\gamma^2(n + 2)]^{-1} \|Du\|^2 + (1 - (\gamma n)[4 + 3\gamma(n+2)]^{-1}) \|Du\|^2.
\]

It is easy to see that for
\[
(20) \quad \gamma < -\frac{2}{n+3}
\]
it takes place (19) with \( c_\gamma = 1 - (\gamma n)[4 + 3\gamma(n+2)]^{-1} > 0 \).

2.3. Monotonicity condition. Let us suppose that \( \lambda > 0 \), \( \nu > 0 \). (From (17) it follows that \( \lambda > 0 \) is implied by the condition (20).) Putting
\[
(21) \quad \overline{V_1 u} = V_1 u (1 + \|\nabla u\|^2)^{\frac{1}{2}}
\]
and denoting by \( I(\varphi) \) the integrand of \( \langle DA(u) \varphi, \varphi \rangle \) we have
\[
(22) \quad I(\varphi) = \|\delta \varphi\|^2 + \lambda \left\{ |\overline{V_1 u} \overline{V_j u} \overline{V_k \varphi} + \overline{V_1 u} \overline{V_j \varphi} \overline{V_k u} + \overline{V_1 \varphi} \overline{V_j u} \overline{V_k u} - 2(\overline{V_u}, \overline{V \varphi}) \right\} + \nu [4 \|\nabla u\|^2 (\overline{V_u}, \overline{V \varphi})^2 + \|\overline{V_u}\|^4 \|\varphi\|^2 - 4 \|\overline{V_u}\|^4 (\overline{V_u}, \overline{V \varphi})^2].
\]
The expression standing by \( \lambda \) in figure brackets can be estimated by means of Hölder inequality as follows:
\[
\{ [\nabla_i u, \nabla_j v, \nabla_k \varphi + \ldots] \} \subseteq \{ i, j, k \in \{ 1, 2, 3 \} \}
\]

\[
(\nabla u, \nabla \varphi)^2 [3 - 6 \| \nabla u \|^2 + 2 \| \nabla u \|^4] \leq (\text{using the fact that } 0 \leq \| \nabla u \| < 1) \leq \| \nabla \varphi \| \left[ 3 \| \nabla u \|^4 \| \nabla \varphi \|^2 + 2 \| \nabla u \|^2 (\nabla u, \nabla \varphi)^2 (1 - \| \nabla u \|^2) \right]^{\frac{1}{2}}.
\]

Using the estimate in (22) and putting
\[
Q^2 = \| \nabla u \|^4 \| \nabla \varphi \|^2 + 2 \| \nabla u \|^2 (\nabla u, \nabla \varphi)^2 (1 - \| \nabla u \|^2)
\]
we obtain
\[
I(\varphi) \geq \| \nabla \varphi \|^2 - \sqrt{3} \lambda Q \| \nabla \varphi \| + \varphi Q^2 \geq \| \nabla \varphi \|^2 (1 - 1\lambda^2). \tag{23}
\]

Let now
\[
4 \rho > 3 \lambda^2 \tag{24}
\]
and let (20) hold. Then \( I(\varphi) \geq c_\varphi^* \| D \varphi \|^2 \) with \( c_\varphi^* > 0 \) and so the monotonicity of the operator \( A \) defined by (4) takes place.

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