Pavel Drábek
Remarks on multiple periodic solutions of nonlinear ordinary differential equations

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 1, 155--160

Persistent URL: http://dml.cz/dmlcz/105984

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1980

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
REMARKS ON MULTIPLE PERIODIC SOLUTIONS OF NONLINEAR
ORDINARY DIFFERENTIAL EQUATIONS
Pavel DRÁBEK

Abstract: We prove the existence and multiplicity of
periodic solutions for nonlinear ordinary differential equa-
tions of the type
\[ u''(x) + g(u(x)) = f(x) \]
under the various conditions upon the function \( g \).

Key words: Nonlinear ordinary differential equations,
periodic problems.

Classification: 34C25

I. Introduction. Our starting point have been the pa-
pers [1],[2]. There are given in [2, Theorem 10] some con-
ditions upon the right hand side \( f \) to obtain at least one
solution of periodic problem
\[
\begin{cases}
  u''(x) + g(u(x)) = f(x) \\
  u(0) = u(T), \ u'(0) = u'(T),
\end{cases}
\]
where \( T \in (0,\pi) \) and \( g \) is \( \tau \)-periodic function on \( \mathbb{R} \) with so-
me \( \tau > 0 \). In this article we present some multiplicity re-
sults for the solvability of (1) using the approach indica-
ted in [1, 26.10] and in [2, Theorem 10], under the assump-
tion that \( g \) is a bounded function, generally not periodic,
with bounded derivative on \( \mathbb{R} \). The presented sufficient
conditions for the solvability of (1) make restriction only
- 155 -
on the $L_1$-norm of the right hand side $f$ in distinction from the conditions presented in [2].

If we put $g(x) = \sin x$, in (1), we obtain the mathematical pendulum equation.

2. Preliminaries. Let $T > 0$ and let us denote $C^0_T$ the Banach space of all continuous and $T$-periodic functions defined on a real line $\mathbb{R}$ with the norm

$$\| u \|_{C^0_T} = \max_{x \in \mathbb{R}} |u(x)|.$$

Let, further, $g$ be a continuous real-valued function such that $g'$ exists almost everywhere in $\mathbb{R}$ and there exist constants $M > 0$, $K > 0$, $t_0 > 0$ such that

$$|g(\xi)| \leq M, \quad |g'(\xi)| \leq K$$

for all $|\xi| \geq t_0$. Assume, in addition, that $g$ is not a constant function.

**Definition.** For $p$, $q$ such that

$$0 < p < q, \sup_{\xi \in \mathbb{R}} g(\xi) = \overline{g}$$

we put $M_{p,q} = M_{p,q}^1 \cup M_{p,q}^2$, where

$$M_{p,q}^1 = \left\{ d \in \mathbb{R} : \exists c_1, c_2 \in \mathbb{R}, 0 \leq c_1 < c_2, \xi \in (c_1, c_2) \Rightarrow g(\xi) > p, \xi \in (-c_2, -c_1) \Rightarrow g(\xi) < q, d \leq c_2 - c_1 \right\},$$

$$M_{p,q}^2 = \left\{ d \in \mathbb{R} : \exists c_1, c_2 \in \mathbb{R}, 0 \leq c_1 < c_2, \xi \in (c_1, c_2) \Rightarrow g(\xi) < q, \xi \in (-c_2, -c_1) \Rightarrow g(\xi) > p, d \leq c_2 - c_1 \right\}.$$

If $\sup M_{p,q} = \infty$ for each $p$, $q$, satisfying (3), then $g$ is called the expansive function.

Assume that the sets $g^{-1}(G)$, $g^{-1}(\overline{G})$ do not contain a
nondegenerate interval. Slightly modifying the proof of Theorem 8 from [2] we obtain

**Lemma 1.** Let \( f \in C^0_T, \quad x_0 \in \mathbb{R} \) and \( K < \pi^2/T^2 \). If \( u_1, u_2 \) are solutions of (1) such that \( u_1(x_0) = u_2(x_0) \).

Then \( u_1 \) and \( u_2 \) coincide on \( \mathbb{R} \).

There is given in [2] a sketch of the proof of

**Lemma 2.** Let \( f \in C^0_T \) and \( K < \pi^2/T^2 \). Then the Dirichlet problem

\[
\begin{aligned}
& u''(x) + g(c + u(x)) = f(x), x \in (0,T), \\
& u(0) = u(T) = 0
\end{aligned}
\]

has a unique solution \( u \in C^2((0,T)) \) for arbitrary \( c \in \mathbb{R} \) (see also [1, Sec. 4.14, 4.19]).

3. Main result

**Theorem.** Let \( f \in C^0_T \) and \( K < \pi^2/T^2 \). Then the problem (1) has at least one \( T \)-periodic solution if

\[
G < q \leq \frac{1}{T} \int_0^T f(x) dx < p < G,
\]

\[
T^2 M + T \int_0^T |f(x)| dx < \sup_M p, q.
\]

**Proof.** Denote by \( \Psi_{c,f} \) the solution of (4) and put

\[
v_{c,f}(x) = c + \Psi_{c,f}(x - kT)
\]

for \( x \in (kT,(k+1)T) \) (\( k \) is an integer). Then \( v_{c,f} \) is a \( T \)-periodic solution of (1) if and only if

\[
\int_0^T g(v_{c,f}(x)) dx = \int_0^T f(x) dx.
\]

Let us define a function \( \Phi_f: \mathbb{R} \rightarrow \mathbb{R} \),
\( \Phi : c \mapsto \int_0^T g(v_c, x(dx)) \).

The Rolle's theorem implies the existence of such \( x_c \in (0, T) \) that \( v'_c(x_c) = 0 \). Using this, we obtain

\[
(5) \quad |v'_c, f'(y)| \leq |\int_{x_c}^y g(c + v'_c, f(x))dx| + |\int_{x_c}^y f(x)dx| \leq TM + \int_0^T |f(x)|dx, \ y \in (0, T), \ c \in R,
\]

\[
(6) \quad |v'_c, f'(y_1) - v'_c, f'(y_2)| \leq \sup_{z \in (0, T)} |v''_c, f'(z)| |y_1 - y_2| \leq T^2M + T \int_0^T |f(x)|dx, \ y_1, y_2 \in (0, T), \ c \in R.
\]

From (6) and from the assumption \( T^2M + T \int_0^T |f(x)|dx < \sup M_{p, q} \) we obtain \( c_1, c_2 \in R \) such that

\[
(7) \quad \Phi(c_1) < Tq \ and \ \Phi(c_2) > Tp.
\]

Let us suppose that \( \lim_{n \to \infty} d_n = d_0 \). Then according to (5), (6) the set \( i\tilde{v}^\infty_{d_n, f} \) satisfies the assumptions of [3, Theorem 1.5.4] and so it is relatively compact in the space of two times continuously differentiable functions on \( (0, T) \). This fact together with Lemma 2 imply that there exists exactly one \( \tilde{v}^\infty_{d_0, f} \) which is the solution of (4) and \( \Phi(d_0) = \lim_{n \to \infty} \Phi(d_n) \). So \( \Phi \) is a continuous function and from (7) we obtain \( c_3 \in (c_1, c_2) \) such that

\[
\Phi(c_3) = \int_0^T f(x) \ dx.
\]

Then \( v_{c_3, f} \) is the solution of (1).

Corollary 1. Let \( f \in C^0_T \), \( K < \pi^2/T^2 \). Suppose, moreover, that \( g \) is an expansive function, \( \sup M_{p, q}^i = \infty, \ i=1,2 \) and \( g^{-1}(\emptyset), g^{-1}(\emptyset) \) are both empty or both infinite. Then the problem (1) has infinitely many distinct solutions if and
only if
\[ g < \frac{1}{T} \int_0^T f(x) dx < \overline{g}, \text{ in the case } g^{-1}(\mathcal{G}) = g^{-1}(\overline{g}) = \emptyset; \]
\[ g < \frac{1}{T} \int_0^T f(x) dx < \overline{g}, f = \mathcal{G}, f = \overline{g}, \text{ in the case } g^{-1}(\mathcal{G}) \neq \emptyset, \]
\[ g^{-1}(\overline{g}) \neq \emptyset. \]

**Proof.** There are \( p, q \in \mathbb{R} \) such that
\[ g < q \leq \frac{1}{T} \int_0^T f(x) dx \leq p < \overline{g}, \]
in the case \( g^{-1}(\mathcal{G}) = \emptyset, g^{-1}(\overline{g}) = \emptyset. \) Because of \( \sup M_i^1 = \infty \), \( i=1,2 \), we obtain \( \{c_n\}_{n=1}^\infty \subset \mathbb{R}, \) \( c_n + c_m \) for \( n \neq m, \)
\[ \Phi_f(c_n) = \int_0^T f(x) dx. \] If \( g^{-1}(\mathcal{G}) \neq \emptyset, g^{-1}(\overline{g}) \neq \emptyset \) then for each \( k_1 \in g^{-1}(\mathcal{G}), \) resp. \( k_2 \in g^{-1}(\overline{g}), \) the function \( u = k_1, \) resp. \( u = k_2, \) is the solution of (1) with \( f = \mathcal{G}, \) resp. \( f = \overline{g}. \) The necessity of the condition follows from the fact that each periodic solution \( u \) of (1) must satisfy
\[ \int_0^T g(u(x)) dx = \int_0^T f(x) dx. \]

**Corollary 2.** Let \( f \in C_{0}^T \), \( K < \pi^2/T^2 \) and, moreover, let \( g \) be a \( \tau \)-periodic function. Then the problem (1) has at least two distinct solutions \( u_1, u_2 \) such that \( |u_i(0)| \leq \tau, \)
\( i=1,2, \) if
\[ -1 - \frac{1}{M_i^1} \int_0^T f(x) dx \leq p < 1 \] and
\[ T^2M + T \int_0^T |f(x)| dx < \sup M_i^1 p. \]

**Proof.** There are fulfilled all the assumptions of Theorem and moreover \( \Phi_f \) is a \( \tau \)-periodic function. There are \( c_1, c_2 \in \mathbb{R}, \) \( c_1 < c_2 < c_1 + \tau \) such that \( \Phi_f(c_1) = \Phi_f(c_1 + \tau) < -Tp, \) \( \Phi_f(c_2) > Tp. \) So we obtain \( c_3 \in (c_1, c_2) \) and \( c_4 \in (c_2, c_1 + \tau) \) such that \( \Phi_f(c_3) = \Phi_f(c_4) = \int_0^T f(x) dx. \)

- 159 -
Remark. From the Corollary 1 it follows that the equation
\[ u''(x) + \sin(u^{2n+1}(x)) = f(x) \]
possesses an infinite number of T-periodic solutions if and only if
\[ -1 < \frac{1}{T} \int_0^T f(x) dx < 1, \quad f = \pm 1. \]

From the Corollary 2 it follows that the mathematical pendulum equation
\[ u''(x) + \sin u(x) = f(x) \]
has at least two distinct T-periodic solutions \( u_1, u_2 \) such that \( |u_i(0)| \leq 2\sigma \), \( i=1,2 \), if
\[ -1 < -p \leq \frac{1}{T} \int_0^T f(x) dx \leq p < 1 \quad \text{and} \quad T^2 + T \int_0^T |f(x)| dx < \pi^2 - 2\arcsin p. \]

References

Katedra matematiky VŠSE
Nejedlého sady 14, 30614 Plzeň
Československo

(Oblatum 2.8. 1979)