Commentationes Mathematicae Universitatis Carolinae

Jarolím Bureš; Jiří Vanžura

A Nijenhuis-type tensor on the quotient of a distribution

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 201--208

Persistent URL: `http://dml.cz/dmlcz/105989`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,2 (1980)

A NIJENHUIS-TYPE TENSOR ON THE QUOTIENT OF A DISTRIBUTION

Jarolím BUREŠ, Jiří VANŽURA

Abstract: Tensor fields of type (1,1) defined on the vector bundle over a differentiable manifold which arises as the quotient of an integrable distribution are studied. For a couple of such tensors a Nijenhuis-type tensor is defined and its applicability to a certain generalized integrability problem is showed.

Key words: Differentiable manifold, integrable distribution, tensor fields.

Classification: 58A30

Let M be a manifold, $\dim M = m$, and let $D \subset E$ be two integrable distributions on M with $\dim D = d$ and $\dim E = e$. We shall denote by F the factorbundle E/D and by π the projection $E \longrightarrow E/D$. Obviously $\dim F = e - d$. We shall consider a tensor field T of type (1,1) on F, i.e. an endomorphism $T: F \longrightarrow F$.

Let X be a vector field on M which is an infinitesimal automorphism of both the distributions D and E, i.e. a vector field satisfying the following two conditions

$$Y_1 \in D \Longrightarrow [X, Y_1] \in D$$
$$Y_2 \in E \Longrightarrow [X, Y_2] \in E$$

With respect to such a vector field X we can define the Lie derivative $L_X A$ of any tensor field A of any type on F. We shall make this definition explicit for tensor fields of types (1,0) and (1.1) only. Let φ_t be the local 1-parameter group of local diffeomorphisms corresponding to the vector field X. Obviously $\varphi_{t*}(D) = D$ and $\varphi_{t*}(E) = E$ so that φ_{t*} induces a local isomorphism (which we denote by the same letter) $\varphi_{t*} : F \to F$. Taking a tensor field $\tilde{Y}$ of type (1,0), i.e. a section of F, we define

$$(L_X \tilde{Y})_x = \lim_{t \to 0} \frac{(\varphi_{t*}^{-1} \tilde{Y})_x - \tilde{Y}_x}{t} .$$

If $Y \in E$ is a vector field such that $\sigma Y = \tilde{Y}$, then it is easy to see that there is $L_X \tilde{Y} = \sigma([X,Y])$.

For a tensor field T of type (1,1) we define first the tensor field $\varphi_{t*}^{-1} T$ by $(\varphi_{t*}^{-1} T)_x(\tilde{v}) = \varphi_{t*}^{-1}(T_{\varphi_t(x)}(\varphi_{t*}\tilde{v}))$ where $\tilde{v} \in F_x$ and F_x denotes the fibre of F over the point $x \in M$. $T_{\varphi_t(x)}$ is the value of T at $\varphi_t(x)$. We set

$$(L_X T)_x = \lim_{t \to 0} \frac{(\varphi_{t*}^{-1} T)_x - T_x}{t} .$$

It is easy to see that for any section Y of F there is

$$L_X(T\tilde{Y}) = (L_X T)(\tilde{Y}) + T(L_X \tilde{Y})$$

We shall now restrict to the study of tensor fields of type (1,1) having a special property. This restriction is quite natural and was motivated by the integrability problem for the tensor fields of type (1,1) on the factorbundle F. Let us recall that a local infinitesimal automorphism of D is a vector field X defined on an open subset $U \subset M$ such that for any vector field $Y \in D$ defined on an open subset $V \subset U$ there is $[X,Y] \in D$ on V. Let us describe the coordinate form of a local infinitesimal automorphism X of D. Because the distribution D is integrable, we can find a chart $(x^1, \dots, x^m)$ such that locally D has the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^d}$. In terms of this chart X has the form

$$X = \sum_{i=1}^{d} a^i(x^1, \dots, x^m)\frac{\partial}{\partial x^i} + \sum_{i=d+1}^{m} a^i(x^{d+1}, \dots, x^m)\frac{\partial}{\partial x^i}.$$

We shall denote by g_x the germ of a vector field or a section of F at the point x.

Definition: We shall say that a tensor field T of type (1,1) on F has the property (P) if the following condition is satisfied for any $x \in M$: If $g_x(\tilde{X})$ is the germ of a local section $\tilde{X}$ of F for which there exists a local infinitesimal automorphism $X \in E$ of D such that $g_x(\tilde{X}) = g_x(\pi X)$, then there exists a local infinitesimal automorphism $Y \in E$ of D such that $g_x(T\tilde{X}) = g_x(\pi Y)$.

Lemma: A tensor field T on F has the property (P) if and only if for any $Y \in D$ there is $L_Y T = 0$.

Proof: (i) Let $L_Y T = 0$ for any local vector field Y belonging to D. Let us take $\tilde{X} = \pi X$. x) For any local vector field $Y \in D$ we have

$$\pi[X', Y] = -\pi[Y, X'] = -L_Y(T\tilde{X}) = -(L_Y T)(\tilde{X}) - T(L_Y\tilde{X}) =$$
$$= -T(L_Y\tilde{X}) = -T(\pi[Y,X]) = T(\pi[X,Y]) = 0$$

because X is an infinitesimal automorphism. This shows that X is again an infinitesimal automorphism.

(ii) Let T satisfy the property (P). For any vector

----

x) with $X \in E$ being a local infinitesimal automorphism of D. We choose any $X' \in E$ such that $T\tilde{X} = \pi X'$.

$\tilde{v} \in F_x$ we can find a section $\tilde{X}$ of F such that $\tilde{X}_x = \tilde{v}$ and $\tilde{X} = \pi X$ with $X \in E$ being a local infinitesimal automorphism of D: This can be immediately seen from the coordinate form of a local infinitesimal automorphism of D. Because T has the property (P), there is a local infinitesimal automorphism X' of D such that $T\tilde{X} = \pi X'$. For any $Y \in D$ defined on an open neighborhood of x we have

$$(L_Y T)_x(\tilde{v}) = ((L_Y T)(\tilde{X}))_x = (L_Y(T\tilde{X}))_x - (T(L_Y\tilde{X}))_x =$$
$$= (\pi[Y,X'])_x - (T(\pi[Y,X]))_x = (T(\pi[X,Y]))_x - (\pi[X',Y])_x = 0$$

which finishes the proof.

Let T_1, T_2 be two tensor fields of type (1,1) defined on F and satisfying (P). We are going to define a tensor field $[T_1,T_2]$ of type (1,2) on F. Let $\tilde{v},\tilde{w} \in F_x$ and let us choose local infinitesimal automorphisms $X,Y \in E$ of D in such a way that $(\pi X)_x = \tilde{v}$ and $(\pi Y)_x = \tilde{w}$. We denote $\tilde{X} = \pi X$, $\tilde{Y} = \pi Y$. Furthermore we choose local infinitesimal automorphisms $X',X'',Y',Y'' \in E$ of D such that $T_1\tilde{X} = \pi X'$, $T_1\tilde{Y} = \pi Y'$, $T_2\tilde{X} = \pi X''$, $T_2\tilde{Y} = \pi Y''$. We set

$$[T_1,T_2]_x(\tilde{v},\tilde{w}) = (\pi[X',Y''])_x + \pi([X'',Y'])_x +$$
$$+ (T_1T_2\pi[X,Y])_x + (T_2T_1\pi[X,Y])_x - (T_1\pi[X'',Y])_x -$$
$$- (T_1\pi[X,Y''])_x - (T_2\pi[X',Y])_x - (T_2\pi[X,Y'])_x.$$

Of course, it is necessary to show that $[T_1,T_2]_x(\tilde{v},\tilde{w})$ does not depend on the choice of X,X',X'' and Y,Y',Y'' with the above properties. Let us notice that because D and E are integrable, we can, in a neighborhood of any point, find a chart $(x^1,\dots,x^m)$ such that $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^d}$ is a local basis of D, and $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^e}$ is a local basis of E. With respect to this chart any local infinitesimal automorphism X of D belonging to E has the form

$$X = \sum_{i=1}^{d} a^i(x^1,\ldots,x^m)\frac{\partial}{\partial x^i} + \sum_{i=d+1}^{e} a^i(x^{d+1},\ldots,x^m)\frac{\partial}{\partial x^i}\ .$$

Using this form of a local infinitesimal automorphism of D it is a matter of the direct calculation to show that $[T_1,T_2]_x(\tilde{v},\tilde{w})$ does not depend on the above mentioned choices. We leave this calculation to the reader.

In the sequel we are going to present one application of the tensor introduced above. We shall consider a tensor field T of type $(1,1)$ on F which satisfies the condition (P). Let us suppose that there exists a matrix $C = (c_\alpha^\beta)$ of the dimension $e-d$ such that for any $x \in M$ we can find a basis $\tilde{v}_1$, $\tilde{v}_{e-d}$ of F_x such that with respect to this basis there is $T_x\tilde{v}_\alpha = \sum_{\beta=1}^{e-d} c_\alpha^\beta \tilde{v}_\beta = 1,\ldots,e-d$. We shall say that the tensor field T is integrable if to any point $x \in M$ there exists a chart $(x^1,\ldots,x^m)$ defined on an open neighborhood U of x and such that

(i) $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^d}$ is a local basis of D on U

(ii) $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^e}$ is a local basis of E on U

(iii) $T(\pi\frac{\partial}{\partial x^{d+}}) = \sum_{\beta=1}^{e-d} c_\alpha^\beta \pi \frac{\partial}{\partial x^{d+\beta}}$ on U.

We shall restrict ourselves to the case where $T^2 = -\,\mathrm{id}$, i.e. to the case of an almost complex structure. Here C is the matrix

$$\begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$$

Of course, much more general results can be easily obtained.

For this purpose compare the results in [1]. We shall prove the following

<u>Proposition</u>: Let T be a tensor field of type (1,1) on F satisfying the condition P and such that $T^2 = -\,\mathrm{id}$. T is integrable if and only if $[T,T] = 0$.

Proof: As usual, it can be easily seen that $[T,T] = 0$ if the tensor field T is integrable. Thus it remains to prove that $[T,T] = 0$ is also a sufficient condition for the integrability of T. To any point $x \in M$ we can find a chart $\varphi = (x^1,\dots,x^m)$ defined on an open neighborhood U of x such that (i) $\varphi(U) = U_1 \times U_2 \times U_3$, where U_1, U_2 and U_3 are open subsets in R^d, R^{e-d} and R^{m-e} respectively, (ii) $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^d}$ is a local basis of D on U, (iii) $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^e}$ is a local basis of E on U. We introduce functions γ_i^j, $i,j = d+1,\dots,m$ by the equality

$$T(\pi \frac{\partial}{\partial x^i}) = \sum_{j=d+1}^{m} \gamma_i^j \,\pi\, \frac{\partial}{\partial x^j} .$$

Because $\frac{\partial}{\partial x^i}$ is a local infinitesimal automorphism of D and T has the property (P), it follows easily that $\sum_{j=d+1}^{m} \gamma_i^j \frac{\partial}{\partial x^j}$ is again a local infinitesimal automorphism of D, i.e. there is

$$\frac{\partial \gamma_i^j}{\partial x^k} = 0 \text{ for } i,j = d+1,\dots,m \text{ and } k = 1,\dots,d.$$

We shall denote by D' the distribution on U generated by the vector fields $\frac{\partial}{\partial x^{d+1}},\dots,\frac{\partial}{\partial x^e}$. The projection π induces an isomorphism $D' \longrightarrow F$, which enables us to transfer the tensor field T from F to D. We denote by T' the tensor field on D obtained in this way. Obviously $T'^2 = -\,\mathrm{id}$. i.e. any leaf of the distribution D is provided with the almost complex structure T. Moreover we have $T' \frac{\partial}{\partial x^i} = \sum_{j=d+1}^{m} \gamma_i^j \frac{\partial}{\partial x^j}$. Because $\frac{\partial \gamma_i^j}{\partial x^k} = 0$ for $i,j = d+1,\dots,m$ and $k = 1,\dots,d$ the almost complex structure T' is (in the obvious sense) the same on all the leaves of D. Taking any leaf of D it can be immediately seen that $[T,T] = 0$ implies the integrability of T on this leaf. This shows that there is a chart $(y^1,\dots$ $\dots,y^m)$ defined on an open neighborhood of x_0 such that

$$y^i = x^i,\ i = 1,\dots,d$$

$$y^i = f^i(x^{d+1},\dots,x^m),\ i = d+1,\dots,e$$

$$y^i = x^i,\ i = e+1,\dots,m$$

and with respect to which (i) $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^d}$ is a local basis of D, (ii) $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^e}$ is a local basis of E, (iii) $T(\pi \frac{\partial}{\partial y^i}) = \sum_{j=d+1}^{e} e_i^j \pi \frac{\partial}{\partial y^j}$, $i,j = d+1,\dots,e$, where (e_i^j) is the matrix $\begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix}$.

Further applications will be subject of a forthcoming paper.

R e f e r e n c e

[1] LEHMANN-LEJEUNE J.: Integrabilité des G-structures définies par une 1-forme 0-deformable à valeurs dans le fibre tangent, Ann. Inst. Fourier, Grenoble 16,2(1966), 329-387.

Matematický ústav
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

Přírodovědecká fakulta
Universita Palackého
Gottwaldova 15, 77500
Olomouc, Československo

(Oblatum 1.11.1979)