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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 293--308

Persistent URL: http://dml.cz/dmlcz/105996

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GENERALIZED FLATNESS AND COHERENCE
Hana JIRASKOVA

Abstract: In this paper flatness and coherence relative to a cohereditary idempotent radical $s$ is studied. Results here obtained are applied to the $\mathcal{M}$-flatness with respect to a pseudoprojective module $\mathcal{M}$.

Key words: Relatively flat modules, relative coherence, preradicals.

Classification: Primary 16A50, 16A52
Secondary 18B40

In what follows, $R$ stands for an associative ring with a unit element and $R$-mod (mod-$R$) denotes the category of all unitary left (right) $R$-modules.

First of all, we shall list several basic definitions from the theory of preradicals.

Recall that a preradical $r$ for $R$-mod is a subfunctor of the identity functor, i.e. $r$ assigns to each module $M$ its submodule $r(M)$ in such a way that every homomorphism of $M$ into $N$ induces a homomorphism of $r(M)$ into $r(N)$ by restriction.

A module $M$ is $r$-torsion if $r(M) = M$ and $r$-torsionfree if $r(M) = 0$. The class of all $r$-torsion ($r$-torsionfree) modules will be denoted by $\mathcal{T}_r$ ($\mathcal{F}_r$).
A preradical \( r \) is said to be
- idempotent if \( r(M) \in \mathcal{T}_r \) for every module \( M \),
- a radical if \( M/r(M) \in \mathcal{F}_r \) for every module \( M \),
- hereditary if for every module \( M \) and every monomorphism \( A \to r(M) \) \( A \in \mathcal{T}_r \),
- cohereditary if for every module \( M \) and every epimorphism \( M/r(M) \to A \) \( A \in \mathcal{F}_r \),
- superhereditary if it is hereditary and \( \mathcal{T}_r \) is closed under direct products,
- centrally splitting if it is cohereditary and \( r(R) \) is a ring direct summand of \( R \).

If \( r \) and \( s \) are preradicals then we write \( r \preceq s \) if \( r(M) \subseteq s(M) \) for all \( M \in R\text{-mod} \).

The idempotent core \( \overline{r} \) of a preradical \( r \) is defined by \( \overline{r}(M) = \bigoplus K \), where \( K \) runs through all \( r \)-torsion submodules \( K \) of \( M \) and the radical closure \( \overline{r} \) is defined by \( \overline{r}(M) = \bigcap L \), where \( L \) runs through all submodules \( L \) of \( M \) with \( M/L \) \( r \)-torsion-free. Further, the hereditary closure \( h(r) \) is defined by \( h(r)(M) = M \cap r(E(M)) \), where \( E(M) \) is an injective hull of a module \( M \) and the cohereditary core \( ch(r) \) by \( ch(r)(M) = = r(R)\cdot M \).

A module \( P \) is called pseudoprojective if for any epimorphism \( f:B \to A \) and any homomorphism \( 0 \neq g:P \to A \), there exist homomorphisms \( h:P \to B \) and \( k:P \to P \) such that \( 0=g\circ k = f\circ h \).

For a module \( M \) let us define \( p_{i|M_1}(N) = \bigoplus \text{Im } f \), \( f \) ranging over all \( f \in \text{Hom}_R(M,N) \). It is easy to see that \( p_{i|M_1} \) is an idempotent preradical. Moreover \( p_{i|M_1} \) is cohereditary if and only if \( M \) is pseudoprojective.

Let \( r \) be a preradical. We say that a submodule \( A \) of a
module B is
- \((r,1)\)-dense in \(B\) if there is a module \(C\) such that \(A \subseteq B \subseteq C\) and \(B/A \subseteq r(C/A)\),
- \((r,2)\)-dense in \(B\) if \(B/A \in \mathcal{T}_r\).

Let \(r\) be a preradical and \(i \in \{1, 2\}\). A module \(Q\) is said to be \((r,i)\)-injective ((i,r)-injective) if for every monomorphism \(f : A \to B\) and every homomorphism \(g : A \to Q\) with \(\text{Im} f \) is \((r,i)\)-dense in \(B\) (\(f(\text{Ker} g)\) is \((r,i)\)-dense in \(B\)) there exists a homomorphism \(h : B \to Q\) such that \(h \circ f = g\).

**Definition 1.** Let \(s\) be a preradical for \(\text{mod}-R\). A module \(R^s Q\) is called \(s\)-flat if \(\text{Tor}_1^R(N, Q) = 0\) for every \(N \in \mathcal{T}_s\).

As it is easy to see, a module \(R^s Q\) is \(s\)-flat if and only if its character module \(|\bar{\mathcal{T}}^s|\) is \((s,2)\)-injective. Since a module is \((\bar{s},2)\)-injective if and only if it is \((s,2)\)-injective, we obtain immediately the following proposition.

**Proposition 2.** If \(s\) is a preradical for \(\text{mod}-R\), then a module \(R^s Q\) is \(s\)-flat if and only if it is \(\bar{s}\)-flat.

The first part of the following proposition is essentially due to R.W. Miller and M.L. Teply [16].

**Proposition 3.** Let \(s\) be a preradical for \(\text{mod}-R\) and \(Q \in R\text{-mod}\). Consider the following conditions.
(i) \(Q\) is \(s\)-flat.
(ii) Given any exact sequence
\[ 0 \to K \to P \to Q \to 0 \]
with \(P\) projective, there is for every \(x \in s(R) \cdot K\) a homomorphism \(f_x : P \to K\) such that \(f_x(x) = x\).
(iii) Given any exact sequence
\[ 0 \to K \to P \to Q \to 0 \]
\[-295-\]
with $P$ projective, there is for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of $s(R) \cdot K$ a homomorphism $f : P \rightarrow K$ such that $f(x_i) = x_i$ for every $i \in \{1, 2, \ldots, n\}$.

(iv) Given any $t_p \in s(R)$, $q_j \in Q$, $r_i, j \in R$, $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$, $p \in \{1, 2, \ldots, q\}$, with $\sum_{i=1}^{m} r_i j = 0$ for each $i \in \{1, 2, \ldots, m\}$, there is $u_k \in Q$ and $b_j, k \in R$, $j \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, t\}$, such that $q_j = \sum_{k=1}^{t} b_j k \cdot u_k$ for $j \in \{1, 2, \ldots, n\}$ and $t_p (\sum_{i=1}^{m} r_i j \cdot b_j k) = 0$ for $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, t\}$, $p \in \{1, 2, \ldots, q\}$.

(v) Every diagram

$$
\begin{array}{c}
0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0 \\
\downarrow g \\
B \rightarrow Q \rightarrow 0
\end{array}
$$

with exact rows, $F$ free, $K$, $F$ finitely generated and $K = s(R) \cdot K$ can be completed by a homomorphism $h : N \rightarrow B$ to a commutative one.

(vi) For every module $N$ for which there is an exact sequence

$$
0 \rightarrow K \rightarrow F \rightarrow N \rightarrow 0
$$

with $F$ free, $K$, $F$ finitely generated and $K = s(R) \cdot K$, the natural homomorphism

$$
\varphi = \varphi_{N, Q} : \text{Hom}_R(N, R) \otimes_R Q \rightarrow \text{Hom}_R(N, Q)
$$

defined via

$$
\varphi(f \otimes q)(n) = f(n) \cdot q, \quad f \in \text{Hom}_R(N, R), \quad q \in Q, \quad n \in N
$$

is an epimorphism.

(vii) Every diagram

$$
\begin{array}{c}
N \\
\downarrow g \\
0 \rightarrow K \rightarrow B \rightarrow Q \rightarrow 0
\end{array}
$$
with exact row, \( K = s(R) \cdot K \) and \( N \) finitely presented can be completed by a homomorphism \( h: N \rightarrow B \) to a commutative one.

(viii) \( Q/(0:s(R)) \cdot R \) is flat in \( R/(0:s(R)) \cdot R \)-mod.

Then (ii) is equivalent to (iii), (iii) is equivalent to (iv) and (v) is equivalent to (vi). If \( s \) is idempotent then (i) implies (ii). Conversely, if \( s \) is cohereditary then (ii) implies (i). Further,

- if \( s(R) \) is finitely generated as a left ideal then (iv) implies (v),
- if \( s(R) \) is finitely generated as a left ideal and \( s(R) \) is idempotent then (v) implies (iv),
- if \( s \) is a cohereditary idempotent radical and \( s(R) \) is finitely generated as a left ideal then (i) is equivalent to (viii),
- if \( s(R) \) is finitely generated as a left ideal and \( R/s(R) \) is flat as a right \( R \)-module then (iv) implies (vii),
- if \( s(R) \) is a ring direct summand in \( R \) then (vii) implies (iv).

Proof: (ii) is equivalent to (iii), (iii) is equivalent to (iv), (i) implies (ii) for \( s \) idempotent and (ii) implies (i) for \( s \) cohereditary. The proof can be led along the same line as in Theorem 2.1 in [16].

(iv) implies (v). Consider the following diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & K & \rightarrow & F & \rightarrow & N & \rightarrow & 0 \\
& & & \downarrow & \epsilon & & & & \\
& & B & \rightarrow & Q & \rightarrow & 0 \\
& & & & \phi & & & & \\
\end{array}
\]

with exact rows, where \( F \) is finitely generated free with a free basis \( \{x_1, x_2, \ldots, x_n\} \), \( K = \bigoplus_{i=1}^m Kx_i \), \( K = s(R) \cdot K \) and
s(R) = \sum_{p=1}^{q} R_p. Set q_j = (g \circ k)(x_j), j \in \{1, 2, \ldots, n\}. Then 
\begin{align*}
  k_i &= \sum_{j=1}^{m_j} r_{ij} \cdot x_j, i \in \{1, 2, \ldots, m\}, \text{ and hence } 0 = (g \circ k)(k_i) = \\
  &= \sum_{j=1}^{m_j} r_{ij} \cdot q_j. \end{align*}
By (iv) there is u_k \in Q and b_j, k \in R, k \in \{1, 2, \ldots, t\}, j \in \{1, 2, \ldots, m\} such that q_j = k_{x_{\alpha_i}} b_j, k \cdot u_k \text{ for } j \in \{1, 2, \ldots, m\} \text{ and } t \in \{1, 2, \ldots, q\} \text{ for } k \in \{1, 2, \ldots, t\} \text{ and } p \in \{1, 2, \ldots, q\}. \quad \text{For } k \in \{1, 2, \ldots, t\} \text{ choose } e_k \in B \text{ such that } f(e_k) = u_k \text{ and define } h: F \to B \text{ by } h(x_j) = \\
  &= \sum_{j=1}^{m_j} b_j, k \cdot e_k. \end{align*}
Then 
\begin{align*}
  (f \circ h)(x_j) &= f(\sum_{j=1}^{m_j} b_j, k \cdot e_k) = \\
  &= f(\sum_{j=1}^{m_j} b_j, k \cdot u_k) = q_j = (g \circ k)(x_j), j \in \{1, 2, \ldots, n\} \text{ and consequently } f \circ h = g \circ k. \end{align*}
Further, if i \in \{1, 2, \ldots, m\}, p \in \{1, 2, \ldots, q\} \text{ then } h(t_{p, k}) = h(\sum_{j=1}^{m_j} t_{p, k} \cdot x_j) = \sum_{j=1}^{m_j} t_{p, k} \cdot x_j = k_{x_{\alpha_i}} b_j, k \cdot e_k = \sum_{j=1}^{m_j} (t_{p, k} \cdot r_{ij}) \cdot b_j, k \cdot e_k = 0. \end{align*}
Thus h(K) = 0 and h induces a homomorphism \( h: N \to B \) such that \( f \circ h = g \). 
(v) implies (ii). Let \( 0 \to K \to F \to F_n \to Q \to 0 \) be an exact sequence, where \( F \) is free with a free basis \( \{x_{\alpha_i}, \alpha_i \in \Lambda\} \). If \( k \in K \) then \( k = \sum_{i=1}^{n} r_i x_{\alpha_i}, r_i \in R, \alpha_i \in \Lambda \). Set \( F_n = \sum_{i=1}^{n} R x_{\alpha_i} \) and define a homomorphism \( g: F_n \to Q \) by \( g(x_{\alpha_i}) = f(x_{\alpha_i}) \) for \( i \in \{1, 2, \ldots, n\} \). It is easy to see that \( g(s(R)k) = 0 \), hence \( g \) induces a homomorphism \( \bar{g}: F_n / s(R)k \to Q \). Now \( F_n / s(R)k \) is finitely presented since \( s(R) \) is finitely generated as a left ideal and \( s(R)^2 = s(R) \) yields \( s(R) / s(R)k = s(R)k \). By (v) there exists a homomorphism \( h: F_n / s(R)k \to F \) such that \( f \circ h = \bar{g} \). Setting \( h(x_{\alpha_i} + s(R)k) = e_i \) for \( i \in \{1, 2, \ldots, n\} \), we have \( f(e_i) = \\
  = (f \circ h)(x_{\alpha_i} + s(R)k) = \bar{g}(x_{\alpha_i} + s(R)k) = f(x_{\alpha_i}), \) hence \( x_{\alpha_i} = -e_i \in K \) for \( i \in \{1, 2, \ldots, n\} \). Let us define \( \varphi: F \to K \) by
\( \varphi(x_{\alpha_i}) = x_{\alpha_i} - e_i \) for \( i \in \{1, 2, \ldots, n\} \) and \( \varphi(x^*) = 0 \) if \( \alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \). For \( t \in \mathfrak{s}(R) \) we have \( t \cdot \sum_{i=1}^{n} r_i e_i = t \cdot \sum_{i=1}^{n} r_i h(x_{\alpha_i}) + s(R)k = h(t \cdot \sum_{i=1}^{n} r_i x_{\alpha_i}) + s(R)k = h(tk) + s(R)k = 0. \) Thus \( \varphi(tk) = \varphi(t \cdot \sum_{i=1}^{n} r_i x_{\alpha_i}) = \sum_{i=1}^{n} r_i (x_{\alpha_i} - e_i) = t \cdot \sum_{i=1}^{n} r_i x_{\alpha_i} = tk. \)

(v) is equivalent to (vi) is routine. (i) is equivalent to (viii). It follows immediately from [16] Corollary 3.4. (iv) implies (vii). Consider the following diagram

\[
\begin{array}{c}
F/s(R)K \\
\downarrow \sigma \\
F/K \\
\downarrow g \\
0 \rightarrow L \rightarrow B \rightarrow Q \rightarrow 0
\end{array}
\]

with exact row, where \( L = s(R)L, \) \( F \) is finitely generated free with a free basis \( \{x_1, x_2, \ldots, x_n\} \), \( K = \sum_{i=1}^{n} R x_i, \) \( s(R) = \sum_{p \in \mathfrak{p}} R p \) and \( \sigma \) is a natural epimorphism. By the same fashion as in (iv) implies (v) we can show that there exists a homomorphism \( h: F/s(R)K \rightarrow B \) such that \( f \circ h = g \circ \sigma \). Let \( r \) be a cohereditary radical in \( R\text{-mod} \) corresponding to \( s(R) \) (i.e. \( r(A) = s(R)A \) for all \( A \in R\text{-mod} \)). By assumption \( L \in \mathcal{T}_r \). Further \( R/s(R) \) is flat as a right \( R \)-module, hence \( r \) is hereditary. Since \( h(K/s(R)K) \subseteq L \), we have \( h(K/s(R)K) \in \mathcal{T}_r \cap \mathcal{F}_r = 0. \) Thus \( h \) induces a homomorphism \( \overline{h}: F/K \rightarrow B \) such that \( f \circ \overline{h} = g. \) (vii) implies (ii). Let \( 0 \rightarrow K \rightarrow F \rightarrow Q \rightarrow 0 \) be an exact sequence, where \( F \) is free with a free basis \( \{x_{\alpha}, \alpha \in \Lambda\} \). By assumption \( s(R) \) is a ring direct summand in \( R \). Thus \( R = s(R) \oplus I \) for some ideal \( I \). Consider the exact sequence

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\[ 0 \rightarrow K/IK \rightarrow F/IK \rightarrow Q \rightarrow 0, \] where \( \overline{f} \) is induced by \( f \). As it is easy to see \( s(R)(K/IK) = K/IK \). Now, if \( k \in K \) then \( k = \sum_{i=1}^{m} r_i \alpha_i, \ r_i \in R, \ \alpha_i \in A \). Set \( F_n = \sum_{i=1}^{m} R \alpha_i \) and define a homomorphism \( g: F_n \rightarrow Q \) via \( g(x_{\alpha_i}) = f(x_{\alpha_i}) \) for \( i \in \{1, 2, \ldots, n\} \). Then \( g(Rk) = 0 \) and \( g \) induces a homomorphism \( \overline{g}: F_n/Rk \rightarrow Q \). Further, \( F_n/Rk \) is finitely presented hence \( \overline{f} \circ h = \overline{g} \) for some homomorphism \( h:F_n/Rk \rightarrow F/IK \) by (vii). Put \( h(x_{\alpha_i} + Rk) = e_i + IK = \overline{e}_i \). As it is easy to see \( x_{\alpha_i} - e_i \in K \) and we can define \( \varphi:F \rightarrow K \) by \( \varphi(x_{\alpha_i}) = x_{\alpha_i} - e_i \) for \( i \in \{1, 2, \ldots, n\} \) and \( \varphi(x_{\alpha_i}) = 0 \) if \( \alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \). We have

\[ h(\sum_{i=1}^{m} r_i \alpha_i + Rk) = h(\sum_{i=1}^{m} r_i \alpha_i) = 0. \] Now, if \( t \in s(R) \) then \( t \cdot \sum_{i=1}^{m} r_i \alpha_i \in s(R)IK = 0 \), hence \( \varphi(tk) = \varphi(\sum_{i=1}^{m} r_i x_{\alpha_i}) = \sum_{i=1}^{m} r_i (x_{\alpha_i} - e_i) = t \cdot \sum_{i=1}^{m} r_i x_{\alpha_i} = tk. \)

**Definition 4.** Let \( s \) be a preradical for \( \text{mod-}R \). A module \( RQ \) satisfying one of the equivalent conditions (ii), (iii) and (iv) of Proposition 3 is said to be weakly \( s \)-flat.

Let \( RQ \) be a flat module. A module \( N_R \) is called \( Q \)-finitely generated (see [6]) if the natural homomorphism \( \psi = \psi_{N_R}: N \otimes_R Q^I \rightarrow (N \otimes_R Q)^I \) defined via \( \psi(n \otimes q)(i) = n \otimes q(i) \) for \( n \in N, \ q \in Q^I, \ i \in I \) is an epimorphism for every set \( I \).

**Theorem 5.** Let \( s \) be a preradical for \( \text{mod-}R \) and \( RQ \) be a flat module. Consider the following conditions

(i) \( Q^I \) is weakly \( s \)-flat for every index set \( I \).
(ii) If \( \{ Q_\alpha, \alpha \in \mathcal{A} \} \) is a family of weakly \( s \)-flat modules, where \( Q_\alpha \in \mathcal{F}_p Q_\alpha \) for every \( \alpha \in \mathcal{A} \) then \( \prod_{\alpha \in \mathcal{A}} Q_\alpha \) is weakly \( s \)-flat.

(iii) \( \text{Hom}_R(P, R) \) is \( Q \)-finitely generated for every module \( P \) for which there exists an exact sequence:

\[
0 \rightarrow K \rightarrow F \rightarrow P \rightarrow 0
\]

with \( F \) free, \( K, F \) finitely generated and \( K = s(R)K \).

(iv) For every finitely generated right ideal \( I \) in \( R \) and an exact sequence \( 0 \rightarrow K \rightarrow F \rightarrow I \rightarrow 0 \) with \( F \) finitely generated free there is a finitely generated submodule \( K' \) of \( F \) such that \( K \otimes R Q \subseteq K' \otimes R Q \) and \( s(R)F(K') = 0 \).

(v) \( (Q/(0:s(R))_P Q)_I \) is flat in \( R/(0:s(R))_P \)-mod for every set \( I \).

Then

- (ii) implies (i), (iv) implies (i),
- if \( s(R) \) is finitely generated as a left ideal then (i) implies (iii) and (i) implies (iv),
- if \( s(R) \) is finitely generated as a left ideal and \( s(R) \) is idempotent then (iii) implies (ii),
- if \( s \) is a cohereditary idempotent radical, \( s(R) \) is finitely generated as a left ideal and \( (0:s(R))_P \) is finitely generated as a right ideal then (i) is equivalent to (v).

**Proof:** (ii) implies (i) trivially.

(i) implies (iii). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_R(P, R) \otimes_R Q^I & \overset{\varphi_{P, Q}^I}{\longrightarrow} & \text{Hom}_R(P, Q^I) \\
\downarrow \varphi & & \downarrow \omega \\
(\text{Hom}_R(P, R) \otimes_R Q)_I & \overset{(\varphi_{P, Q})^I}{\longrightarrow} & (\text{Hom}_R(P, Q))_I
\end{array}
\]
where \( \omega \) is the natural isomorphism and \( \varphi \) is defined as in Proposition 3 (vi). Now \( (\varphi P, Q)^I \) is an isomorphism (see \[14\]), since \( Q \) is flat and \( P \) is finitely presented. Further, \( \varphi P, Q)^I \) is an epimorphism by Proposition 3 (vi). Hence \( \psi \) is an epimorphism and consequently \( \text{Hom}_R(P, R) \) is \( Q \)-finitely generated.

(iii) implies (ii). For \( N \in \text{mod}-R \) the class of all \( M \in \text{R-mod} \)
for which \( N \) is \( M \)-finitely generated is closed under the formation of direct sums of copies of \( M \). Now if \( Q_\alpha \in \mathcal{T}_{P\pi Q}^\beta \), \( \alpha \in \Delta \) then there exist a set \( I \) and epimorphisms \( f_\alpha : Q(I) \to Q_\alpha \), \( \alpha \in \Delta \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_R(P, R) \otimes_R (Q(I))_A & \xrightarrow{\psi} & (\text{Hom}_R(P, R) \otimes_R Q(I))_A \\
1 \otimes \prod_{\alpha \in \Delta} Q_\alpha & \xrightarrow{\psi_1} & \prod_{\alpha \in \Delta} (\text{Hom}_R(P, R) \otimes_R Q) \\
\end{array}
\]

where \( \psi_1(f \otimes q)(\alpha) = f \otimes q(\alpha) \) for \( f \in \text{Hom}_R(P, R) \), \( q \in \beta \otimes A Q_\beta \), \( \alpha \in \Delta \). Then \( \psi \) is an epimorphism since \( \text{Hom}_R(P, R) \) is \( Q(I) \)-finitely generated, hence \( \psi_1 \) is an epimorphism. Now consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_R(P, R) \otimes_R \prod_{\alpha \in \Delta} Q_\alpha & \xrightarrow{\varphi P, \prod_{\alpha \in \Delta} Q_\alpha} & \text{Hom}_R(P, \prod_{\alpha \in \Delta} Q_\alpha) \\
\prod_{\alpha \in \Delta} (\text{Hom}_R(P, R) \otimes_R Q_\alpha) & \xrightarrow{\omega} & \prod_{\alpha \in \Delta} \text{Hom}_R(P, Q_\alpha) \\
\end{array}
\]

where \( \omega \) is the natural isomorphism and \( \varphi \) is defined as in Proposition 3 (vi). Then \( \varphi P, Q_\alpha \) is an epimorphism for every \( \alpha \in \Delta \) by Proposition 3 (vi). Hence \( \varphi P, \prod_{\alpha \in \Delta} Q_\alpha \) is an epimorphism and consequently \( \prod_{\alpha \in \Delta} Q_\alpha \) is weakly \( s \)-flat by Proposition-
on 3.

(i) implies (iv). Suppose $I = \sum_{i=1}^{m} a_i R$ and $0 \rightarrow K \rightarrow F \rightarrow I \rightarrow 0$ is an exact sequence, where $F$ is free with a free basis $\{x_1, x_2, \ldots, x_n\}$ and $f(x_i) = a_i$ for $i \in \{1, 2, \ldots, n\}$. Now, if $k \in K$ then $k = \sum_{i=1}^{m} x_i r_i(k)$ for some $r_i(k) \in R$, $i \in \{1, 2, \ldots, n\}$. Let us define $q_i \in Q^{K \otimes Q}$ by $q_i(k, q) = r_i(k)q$ for $q \in Q$, $k \in K$, $i \in \{1, 2, \ldots, n\}$. Since $0 = \sum_{i=1}^{m} a_i r_i(k)q$ for every $k \in K$ and $q \in Q$, we have $\sum_{i=1}^{m} a_i q_i = 0$ in $Q^{K \otimes Q}$. Let $s(R) = \sum_{p \in \mathcal{P}} R_p$. Then there exist $u_j \in Q^{K \otimes Q}$ and $b_i, j \in R$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$ such that $q_i = \sum_{p \in \mathcal{P}} b_i, j u_j$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Set $k'_j = \sum_{i=1}^{m} x_i b_i, j$, $j \in \{1, 2, \ldots, m\}$ and $K' = \sum_{j=1}^{m} k'_j R$. For $k \in K$, $q \in Q$ we have $k \otimes q = \sum_{i=1}^{m} x_i r_i(k) \otimes q = \sum_{i=1}^{m} x_i \otimes r_i(k)q = \sum_{i=1}^{m} x_i \otimes \sum_{j=1}^{m} b_i, j u_j(k, q) = \sum_{j=1}^{m} k'_j \otimes \sum_{j=1}^{m} b_i, j u_j(k, q) = \sum_{j=1}^{m} k'_j \otimes u_j(k, q) \in K' \otimes_{R} Q$. Further, $t_p f(k'_j) = t_p f(\sum_{i=1}^{m} x_i b_i, j) = t_p (\sum_{i=1}^{m} a_i b_i, j) = 0$ and consequently $s(R)f(K') = 0$.

(iv) implies (i). Suppose $I$ is an arbitrary set, $t_w \in s(R)$, $a_i \in Q^I$, $r_i \in R$, $i \in \{1, 2, \ldots, n\}$, $w \in \{1, 2, \ldots, z\}$ and $\sum_{i=1}^{m} r_i a_i = 0$. Set $J = \sum_{i=1}^{m} r_i R$ and consider an exact sequence $0 \rightarrow K \rightarrow F \rightarrow J \rightarrow 0$, where $F$ is free with a free basis $\{x_1, x_2, \ldots, x_n\}$ and $f(x_i) = r_i$ for $i \in \{1, 2, \ldots, n\}$. Then there exists a finitely generated submodule $K' = \sum_{p \in \mathcal{P}} k'_p R$ of $F$ such that $K' \otimes_{R} Q \in K' \otimes_{R} Q$ and $s(R)f(K') = 0$. Now $Q$ is flat and $\sum_{i=1}^{m} r_i a_i (\alpha) = 0$ for every $\alpha \in I$, hence there exist $v_j(\alpha) \in Q$ and $b_i, j(\alpha) \in R$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$,
\[ \alpha \in A \text{ such that } a_i(\alpha) = \frac{\alpha_i}{\alpha_j}, \text{ for } i = 1, 2, \ldots, n, \]
\[ a \in A \text{ and } \sum_{i=1}^{m} r_i b_{i,j}(\alpha) = 0, \text{ for } j = 1, 2, \ldots, m. \]
Let us denote
\[ u_j(\alpha) = \sum_{i=1}^{n} x_i b_{i,j}(\alpha), \text{ for } j = 1, 2, \ldots, m. \]
Then \( f(u_j(\alpha)) = \sum_{i=1}^{n} r_i b_{i,j}(\alpha) = 0 \) for \( j = 1, 2, \ldots, m, \alpha \in A. \)
Thus \( u_j(\alpha) \in K. \)
Hence \( \sum_{i=1}^{n} u_j(\alpha) \otimes v_j(\alpha) = \sum_{i=1}^{m} k_p \otimes w_p(\alpha) \) for some \( w_p(\alpha) \in Q, \)
\( p \in \{1, 2, \ldots, q\}, \alpha \in A. \)
Further, \( k_p = \sum_{i=1}^{n} x_i d_i, p, d_i \in R, \)
\( i = 1, 2, \ldots, n, p \in \{1, 2, \ldots, q\}. \)
Thus \( \sum_{i=1}^{n} x_i \otimes a_i(\alpha) = \frac{\alpha_i}{\alpha_j}, x_i \otimes \)
\( \sum_{i=1}^{m} b_{i,j}(\alpha) v_j(\alpha) = \sum_{i=1}^{m} (\sum_{i=1}^{n} x_i b_{i,j}(\alpha)) \otimes v_j(\alpha) = \sum_{i=1}^{m} (\sum_{i=1}^{n} x_i d_i, p) \otimes w_p(\alpha) = \frac{\alpha_i}{\alpha_j} x_i \otimes \)
\( \sum_{i=1}^{m} d_i, p, w_p(\alpha). \)
Hence \( a_i(\alpha) = \frac{\alpha_i}{\alpha_j} d_i, p, w_p(\alpha) \) for \( i = 1, 2, \ldots, n, \alpha \in A \) and consequently \( a_i = \frac{\alpha_i}{\alpha_j} d_i, p, w_p, i \in \{1, 2, \ldots, n\}. \)
We have \( t_w(\sum_{i=1}^{n} r_i d_i, p) = t_w(k_p) \in s(R)f(K') = 0, \)
\( w \in \{1, 2, \ldots, z\}, p \in \{1, 2, \ldots, q\}. \)
Hence \( Q^+ \) is weakly \( s \)-flat by Proposition 3.

(i) is equivalent to (v). It immediately follows from Proposition 3 (viii).

**Corollary 6.** Let \( s \) be a preradical for \( \text{mod}-R. \) Consider the following conditions:

(i) \( R^R \) is weakly \( s \)-flat for every set \( I. \)

(ii) Weakly \( s \)-flat modules are closed under direct products.

(iii) \( \text{Hom}_R(P, R) \) is finitely generated for every module \( P \) for which there exists an exact sequence \( 0 \rightarrow K \rightarrow F \rightarrow P \rightarrow 0 \)
with \( F \) free, \( K, F \) finitely generated and \( K = s(R)K. \)

(iv) For every finitely generated right ideal \( I \) in \( R \) and an exact sequence \( 0 \rightarrow K \rightarrow F \overset{f}{\rightarrow} I \rightarrow 0 \) with \( F \) finitely generated free there is a finitely generated submodule \( K' \) of \( F \)
such that \( K \subseteq K' \) and \( s(R)f(K') = 0. \)
(v) $R/(0:s(R))_R$ is a right coherent ring.

Then

- (ii) implies (i), (iv) implies (i),
- if $s(R)$ is finitely generated as a left ideal then (i) implies (iii) and (iv),
- if $s(R)$ is finitely generated as a left ideal and $s(R)$ is idempotent then (iii) implies (ii),
- if $s$ is an idempotent cohereditary radical, $s(R)$ is finitely generated as a left ideal and $(0:s(R))_R$ is finitely generated as a right ideal then (i) is equivalent to (v).

Let $M \in \text{mod-}R$. We recall that a module $RQ$ is said to be $M$-flat if $\oplus_R Q$ is exact on all exact sequences of the form $0 \to A \to M \to C \to 0$.

**Proposition 7.** Let $M \in \text{mod-}R$ be a pseudoprojective module. Then a module $RQ$ is $M$-flat if and only if it is $p_{\{M\}}M$-flat.

**Proof:** First of all, $p_{\{M\}}M$ is an idempotent cohereditary radical for $M$ pseudoprojective. Further, $Q$ is $M$-flat if and only if its character module $Q^e_R$ is $M$-injective. Now $Q^e_R$ is $M$-injective iff it is $(1,p_{\{M\}}2)$-injective. We have $Q^e_R$ is $(1,p_{\{M\}}2)$-injective iff it is $(p_{\{M\}}2)$-injective since $p_{\{M\}}2$ is idempotent cohereditary. Finally $Q^e_R$ is $(p_{\{M\}}2)$-injective iff $RQ$ is $p_{\{M\}}2$-flat.

Now, if we apply Proposition 3, Theorem 5 and Corollary 6 to the $p_{\{M\}}M$-flatness with respect to a pseudoprojective module $M$, we obtain a characterization of $M$-flat modules and a characterization of rings for which a direct product of $M$-flat modules is $M$-flat.

**Proposition 8.** For a preradical $r$ for $R$-mod let us de-
fine the following classes of modules

\( \mathcal{A}_r = \{ X \in \text{mod-} R; X \otimes_R T = 0 \text{ for each } T \in \mathcal{T}_r \} \),
\( \mathcal{B}_r = \{ X \in \text{mod-} R; X \otimes_R r(A) = 0 \text{ for each } A \in R\text{-mod} \} \),
\( \mathcal{C}_r = \{ X \in \text{mod-} R; X \otimes_R Y = 0 \text{ for each } A \in R\text{-mod and } Y \in r(A) \} \),
\( \mathcal{D}_r = \{ X \in \text{mod-} R; X \otimes_R r(P) = 0 \text{ for each projective } P \in R\text{-mod} \},
\( \mathcal{E}_r = \{ X \in \text{mod-} R; X \otimes_R Y = 0 \text{ for each projective } P \in R\text{-mod and } Y \in r(P) \} \).

It is easy to see that \( \mathcal{A}_r, \mathcal{B}_r, \mathcal{C}_r, \mathcal{D}_r \) and \( \mathcal{E}_r \) are torsion classes. Let us denote \( A_r, B_r, C_r, D_r \) and \( E_r \) idempotent radicals corresponding to them. Then

- \( \mathcal{A}_r = \mathcal{B}_r = \mathcal{D}_r = \mathcal{C}_r = \mathcal{E}_r = 0 \),
- \( \mathcal{A}_r = \mathcal{B}_r = \mathcal{E}_r = 0 \),
- \( \mathcal{A}_r = \mathcal{B}_r = \mathcal{D}_r = \mathcal{C}_r = 0 \),
- if \( h(r) \) is superhereditary then \( C_r \) is cohereditary,
- if \( h(ch(r)) \) is a superhereditary radical then \( E_r \) is cohereditary and \( E_r(R) = C_r(ch(r))(R) = (0; r(R))_L \).

**Proof:** Easy.

As consequences of Propositions 3, 5, 6 and 8 we obtain for \( h(r) \) superhereditary a characterization of \( C_r \)-flat modules and of rings for which a direct product of \( C_r \)-flat modules is \( C_r \)-flat.

**References**


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(Oblatum 18.12. 1979)