Ivan Kolář
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ON THE AUTOMORPHISMS OF PRINCIPAL FIBRE BUNDLES
Ivan KOLÁŘ

Abstract: Using Palais-Terng theorem on natural bundles, we determine all smooth principal fibre bundles with the property that the group of all automorphisms can be expressed as a semi-direct product of a prescribed type.

Key words: Principal fibre bundle, natural bundle, jet, gauge transformation.

Classification: 58A20

This research was inspired by a discussion with Prof. A. Trautman and by his paper on gauge transformations [3].

Consider a principal fibre bundle \( \pi: P \to M \) with structure group \( G \). Let \( \text{Aut} P \) be the group of all automorphisms of \( P \). We have an exact sequence

\[
0 \to \text{Aut}_V P \to \text{Aut} P \to \text{Diff} M,
\]

where \( \text{Aut}_V P \) means the group of all vertical automorphisms of \( P \), [3]. An interesting problem is: under what conditions \( \text{Aut} P \) can be expressed as a semi-direct product of \( \text{Aut}_V P \) and \( \text{Diff} M \)? In general, given an exact sequence of groups

\[
0 \to A \to B \to C \to 0,
\]

\( B \) can be expressed as a semi-direct product of \( A \) and \( C \) iff there exists a splitting of (2). We shall determine all \( P \)
such that there is a splitting of (1) with the following "local" property. Denoting by $\text{LDiffM}$ the pseudogroup of all local diffeomorphisms of $M$ and by $\text{IAutP}$ the pseudogroup of all local automorphisms of $P$, we shall assume that the splitting $\text{DiffM} \to \text{AutP}$ is the restriction of a splitting $S: \text{LDiffM} \to \text{IAutP}$ defined on the whole pseudogroup $\text{LDiffM}$.

Such a splitting $S$ is a lifting functor on $P$ in the sense of A. Nijenhuis [1], which endows $P$ with the structure of a natural bundle. According to a recent theorem by R. S. Palais and C.L. Terng (and a related result by D.B.A. Epstein and W. Thurston) [2], any natural bundle has finite order. Given an $r$-th order natural bundle $E \to M$ with lifting functor $F$ and an element $c \in M$, any element $X$ of the group $L^r_c M$ of all invertible isotropic $r$-jets on $M$ at $c$ determines a diffeomorphism $FX: E_c \to E_c$. The assignment $X \mapsto FX$ is a smooth left action of $L^r_c M$ on $E_c$ [2]. Conversely, given a smooth left action $\varphi$ of the group $L^r_n = L^r_c \mathbb{R}^n$ on a manifold $Q$, $n = \dim M$, we can construct the associated fibre bundle $Q^r_M = (M, Q, L^r_n, H^r M)$, where $H^r M$ means the $r$-th order frame bundle of $M$. The bundle $Q^r_M$ is natural with respect to the following lifting functor $F$. Any local diffeomorphism $f: U \to V$ on $M$ is prolonged into a principal bundle isomorphism $H^r f: H^r U \to H^r V$ and we define $Ff: p^{-1}(U) \to p^{-1}(V)$ by $Ff(u, q) = (H^r f(u), q)$, where $p$ denotes the bundle projection of $Q^r_M$.

In our case, $S$ is a functor into the category of principal fibre bundles, so that $S_X: P_c \to P_c$ satisfies $S_X(ug) = (S_X(u))g$. If we fix an element $u \in P_c$, we obtain a map $S_u: L^r_c M \to G$ defined by
SX(u) = uS_uX.

As S(YX)(u) = SY(uS_uX) = u(S_uY)(S_uX), S_u is a group homomorphism. Conversely, let G be a Lie group and \( \varphi: \mathfrak{l}_n^G \to G \) an analytic homomorphism. Then \( (X,g) \mapsto \mathcal{E}(X)g \) is a left action of \( \mathfrak{l}_n^G \) on G and we can construct the associated fibre bundle \( P = (M, G, \mathfrak{l}_n^G, H^M) \). Any element of P being an equivalence class of the equivalence relation \( (u,g) \sim (uX, \mathcal{E}(X^{-1})g) \), \( u \in P \), \( g \in G \), \( X \in \mathfrak{l}_n^G \), we have a well-defined right action \( P \times G \to P \), \( ((u,g), h) \mapsto (u, gh) \). One verifies directly that \( P(M, G) \) is a principal fibre bundle and the induced lifting functor \( S \) is a splitting \( S: \text{LDiff}^M \to \text{LAut}^P \). Thus, we have deduced

**Theorem.** If \( P \) is a principal fibre bundle such that there exists a splitting \( S: \text{LDiff}^M \to \text{LAut}^P \), then there is an integer \( r \) and an analytic homomorphism \( \varphi: \mathfrak{l}_n^r \to G \) such that \( P \) coincides with the corresponding bundle \( (M, G, \mathfrak{l}_n^r, H^M) \) and \( S \) is constructed as above.

**References**


Matematický ústav ČSAV
Janáčkovo nám.2a, 66295 Brno
Československo

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