Commentationes Mathematicae Universitatis Carolinae

Zdeněk Dostál

# A note on estimate of the spectral radius of symmetric matrices

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 21 (1980), No. 2, 333--340

Persistent URL: http://dml.cz/dmlcz/106000

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
21,2 (1980)

# A NOTE ON ESTIMATE OF THE SPECTRAL RADIUS OF SYMMETRIC MATRICES

Zdeněk DOSTÁL

Abstract: The paper presents the quantitative refinement of the spectral radius formula for the $l_\infty$-norm and symmetric matrices.

Key words: Spectral radius, norm of iterates, symmetric matrices.

Classification: 15A12

---

1. Introduction. An information about the spectral radius of a given matrix is often useful for the solution of practical problems. Since it is difficult to compute the spectral radius, it is natural to look for some other quantity which is easier evaluable and which can give us some significant information about the spectral radius. The well known spectral radius formula suggests that such a quantity may be some norm of matrix powers; we shall restrict our attention to the $l_\infty$-norm because of its simplicity.

The first striking result of such a kind was obtained in 1957 by J. Mařík and V. Pták [3], who have proved that for each nxn complex valued matrix A the equation $|A|_\infty^{n^2-n+1} = |A^{n^2-n+1}|_\infty$ implies $|A|_6 = |A|_\infty$. Later, at a suggestion of Professor V. Pták, the present author [1,2] proved several results about relations between the $l_\infty$-norm of matrix powers and the spectral radius. For further references see [4].

The aim of this note is to clear up the relations between the spectral radius and the $l_\infty$-norm of powers of symmetric matrices.

2. Notation and preliminaries. Let n be an arbitrary but fixed positive integer, let $M_n$ denote the algebra of all nxn complex valued matrices, and let I denote the identity matrix in $M_n$. If $A \in M_n$, then we denote by $A^*$ the adjoint (conjugate transpose) of A, by $\sigma(A)$ the spectrum of A, and by $|A|_\sigma$ the spectral radius of A. A Hermitian symmetric matrix $P \in M_n$ is said to be an orthogonal projector if $P^2 =$ $= P$, and a symmetric involution if $P^2 = I$; instead of saying that P is positive semidefinite we shall write $P \geq 0$.

We shall denote by $B_n$ the complex n-dimensional linear space of all nx1 complex valued matrices. The $l_\infty$-norm on the space $B_n$ is defined by the formula

$$|(x_i)|_\infty = \max_i |x_i| .$$

If $x \in B_n$, then the conjugate transpose of x will be denoted by $x^*$, it is a row vector. The operator norm of the matrix $(a_{i,j}) \in M_n$, considered as an operator on the Banach space $(B_n, |\cdot|_\infty)$ turns out to be

$$|(a_{i,j})|_\infty = \max_i \sum_j |a_{i,j}| .$$

If K is a set of real numbers, then we shall write

$$DK = \{A \in M_n : A^* = A,\ \sigma(A) \subset K\}.$$

If K is a compact set of real numbers, then it may be proved that DK is compact. We shall use the following lemmas:

2.1. Let a, b be real numbers, a = b. Put $[a,b] = \{x : a \leqq x \leqq b\}$. Then $D[a,b]$ is the convex hull of $D\{a,b\}$.

Proof: Let $A \in D[a,b]$, let $\lambda_1 \leqq \lambda_2 \leqq \ldots \leqq \lambda_n$ be the eigenvalues of A and let

$$A = \sum_{i=1}^{n} \lambda_i P_i$$

be the spectral decomposition of A. For i = 2,3,...,n, put

$$Q_i = \sum_{j=i}^{n} P_j,$$

thus

$$A = \lambda_1 I + \sum_{i=2}^{n} (\lambda_i - \lambda_{i-1}) Q_i = \frac{\lambda_1 - a}{b - a} \cdot bI +$$

$$+ \sum_{i=2}^{n} \frac{\lambda_i - \lambda_{i-1}}{b - a} [(b-a)Q_i + aI] + \frac{b - \lambda_n}{b - a} \cdot aI.$$

To finish the proof, it is enough to note that the last expression is the convex combination of matrices aI, bI and $(b-a)Q_i + aI$ from $D\{a,b\}$.

2.2. Let $P \in M_n$ be an orthogonal projector, let $a,b \in B_n$, $o \neq b$ and let Pa = b. Then there are complex numbers $q_1, \ldots, q_n$ such that the matrix $Q = (q_i^* q_j)$ is an orthogonal projector of the rank 1, Qa = b and $\sum_{i=1}^{n} |q_i|^2 = 1$.

Proof: Let P, a and $b = (b_i)$ satisfy the assumptions. Put $|b| = (\sum_{i=1}^{n} |b_i|^2)^{1/2}$, $q_i = b_i^*/|b|$ and $Q = (q_i^* q_j)$. It is easy to verify directly that $Q^2 = Q$, $Q^* = Q$, $\sum_{i=1}^{n} |q_i|^2 = 1$, Qb = b, and rank Q = 1. Hence Q is the orthogonal projector onto the Span{b}. Since Pa = b and Pb = b, the vectors b and b-a are necessarily orthogonal and we have

$$Q(b-a) = o, \; Qa = Qb = b.$$

2.3. Let $\mathcal{Q}$, $\mathcal{H}$ be real quadratic forms defined by

$$\mathcal{Q}(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_1 x_i, \; \mathcal{H}(x_1,\ldots,x_n) = \sum_{i=2}^{n} x_1 x_2 - x_1^2.$$

Then

$$\max_{x_i^2=1} \mathcal{Q}(x_1,\ldots,x_n) = \max_{\substack{\Sigma x_i^2=1 \\ x_i \geq 0}} \mathcal{Q}(x_1,\ldots,x_n) = 1/2 + \sqrt{n}/2,$$

and

$$\max_{x_i^2=1} \mathcal{H}(x_1,\ldots,x_n) = \max_{\substack{\Sigma x_i^2=1 \\ x_i \geq 0}} \mathcal{H}(x_1,\ldots,x_n) = -1/2 + \sqrt{n}/2.$$

Proof: The extreme values of the form $\mathcal{Q}$ are equal to the extreme eigenvalues of the matrix

$$Q = \begin{bmatrix} 1 & 1/2 & 1/2 & \ldots & 1/2 \\ 1/2 & 0 & 0 & \ldots & 0 \\ 1/2 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 1/2 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

of the form $\mathcal{Q}$. Computing the trace of Q and the sum of all diagonal minors of rank two, we obtain the characteristic equation

$$\lambda^n - \lambda^{n-1} - \frac{n-1}{4} \lambda^{n-2} = 0$$

of Q; its extreme solutions are

$$\lambda_{max} = 1/2 + \sqrt{n}/2, \; \lambda_{min} = 1/2 - \sqrt{n}/2.$$

In the same way we derive the characteristic equation

$$\lambda^n + \lambda^{n-1} - \frac{n-1}{4} \lambda^{n-2} = 0$$

of the matrix of the form $\mathcal{H}$, its extreme solutions are

$$\lambda_{max} = -1/2 + \sqrt{n}/2, \; \lambda_{min} = -1/2 - \sqrt{n}/2.$$

Since both $\mathcal{Q}(x_1,\dots,x_n) \leqq \mathcal{Q}(|x_1|,\dots,|x_n|)$ and $\mathcal{H}(x_1,\dots,x_n) \leqq \mathcal{H}(|x_1|,\dots,|x_n|)$, the maxima may be attained with nonnegative $x_1,\dots,x_n$.

### 3. Estimates

3.1. Theorem. Let $A \in M_n$, $A \geqq 0$ and $k > 0$. Then

$$|A|_\sigma \geqq (1/2 + \sqrt{n}/2)^{-1/k} |A^k|_\infty^{1/k} \tag{1}$$

and the bound is the best possible one.

Proof: Let us compute

$$K_P = \max\{|A|_\infty : |A|_\sigma \leqq 1,\ A \in M_n,\ A \geqq 0\} =$$
$$= \max\{|A|_\infty : A \in D[0,1]\}.$$

Since the compact set $D[0,1]$ is by the lemma 2.1 the convex hull of $D\{0,1\}$ and since the function $A \mapsto |A|_\infty$ is convex, the maximum $K_P$ exists and is attained by the $l_\infty$-norm of some orthogonal projector from $D\{0,1\}$.

Suppose that $P$ is such an orthogonal projector that $|P|_\infty = K_P$ and let $a \in B_n$, $|a|_\infty = 1$, $|Pa|_\infty = K_P$. We have proved in the lemma 2.2 that there is an orthogonal projector

$$Q = (q_i^* q_j),\quad \sum_{i=1}^{n} |q_i|^2 = 1, \tag{2}$$

such that $Pa = Qa$. Further,

$$K_P \geqq |Q|_\infty \geqq |Qa|_\infty = |Pa|_\infty = K_P,$$

so that the maximum is attained by some orthogonal projector of the form (2). Since each matrix of the form (2) is an orthogonal projector, we can write

$$K_P = \max_i \max_{\sum |q_j|^2 = 1} \sum_{j=1}^{n} |q_i^* q_j| = \max_{\substack{p_i^2 = 1 \\ p_i \geqq 0}} \sum_{i=1}^{n} p_1 p_i = 1/2 + \sqrt{n}/2.$$

Since both the spectral radius and the norm are homogeneous functions, we have proved that $A \in M_n$, $A \geqq 0$ implies

$$|A|_{\infty} \leqq (1/2 + \sqrt{n}/2)|A|_{\sigma} .$$

If $A \geqq 0$ and $k > 0$, then $A^k \geqq 0$, hence

$$|A^k|_{\infty} \leqq (1/2 + \sqrt{n}/2)|A^k|_{\sigma} = (1/2 + \sqrt{n}/2)|A|_{\sigma}^k ,$$

which is equivalent to (1). Equality is attained by the scalar multiples of the orthogonal projectors with maximal $l_{\infty}$-norm.

3.2. <u>Theorem.</u> Let $A \in M_n$, $A^* = A$. If k is an odd natural number, then

(3) $$|A|_{\sigma} \geqq n^{-1/2k}|A^k|_{\infty}^{1/k} ,$$

if k is even, then

(4) $$|A|_{\sigma} \geqq (1/2 + \sqrt{n}/2)^{-1/k}|A^k|_{\infty}^{1/k} .$$

These bounds cannot be improved.

Proof: Let us compute

$$K_H = \max\{|A|_{\infty} : |A|_{\sigma} \leqq 1,\ A \in M_n,\ A^* = A\} =$$
$$= \max\{|A|_{\infty} : A \in D[-1,1]\} .$$

The compact set $D[-1,1]$ being the convex hull of $D\{-1,1\}$, the maximum $K_H$ exists and is attained by the $l_{\infty}$-norm of some symmetric involution from $D\{-1,1\}$. Since the mapping

$$D\{0,1\} \ni P \mapsto 2P-I \in D\{-1,1\}$$

is a 1-1 mapping of the set of all orthogonal projectors onto the set of all symmetric involutions, we can write

$$K_H = \max\{|2P-I|_{\infty} : P \in D\{0,1\}\} .$$

Now suppose that P is such an orthogonal projector that

$|2P-I|_\infty = K_H$ and let $a \in B_n$, $|a|_\infty = 1$, $|(2P-I)a|_\infty = K_H$. We have proved in the lemma 2.2 that there is an orthogonal projector $Q = (q_i^* q_j) \in D\{0,1\}$, $\sum_{i=1}^{n} |q_i|^2 = 1$ such that $Pa = Qa$. Thus

$$K_H \geqq |2Q-I|_\infty \geqq |(2Qa-a)|_\infty = |2Pa-a|_\infty = K_H$$

and

$$K_H = \max_i \max_{|q_j|^2=1} \left( \sum_{\substack{j=1 \\ j \neq 1}}^{n} 2|q_i^* q_j| + |2|q_i|^2 - 1| \right) =$$

$$= \max\{\max\{(\sum_{i=1}^{n} 2p_1 p_i - 1) : \sum_{i=1}^{n} p_i^2 = 1,\ p_i \geq 0\},$$

$$\max\{(\sum_{i=}^{n} 2p_1 p_i - 2p_1^2 + 1) : \sum_{i=1}^{n} p_i^2 = 1,\ p_i \geq 0\}\}.$$

The two inner maxima in the last term are both equal to $\sqrt{n}$ by the lemma 2.3. Hence $K_H = \sqrt{n}$; both the spectral radius and the norm being homogeneous functions, we have proved that $A \in M_n$, $A^* = A$ implies

$$|A|_\infty \leqq \sqrt{n}|A|_\sigma .$$

If $A^* = A$ and $k > 0$, then also $A^{k*} = A^k$, thus

$$|A^k|_\infty \leqq \sqrt{n}|A^k|_\sigma = \sqrt{n}|A|_\sigma^k ,$$

which is equivalent to (3). If P is some extremal involution, that is

$$P^* = P,\ P^2 = I,\ |P|_\infty = K_H,$$

if k is an odd number, and if $\lambda$ is some real number, then $|\lambda P|_\sigma = |\lambda|$, $P^k = P$ and

$$|(\lambda P)^k|_\infty = |\lambda^k P|_\infty = \sqrt{n}\ |\lambda|^k = \sqrt{n}\ |\lambda P|_\sigma^k ,$$

which shows that the bound (3) is the best possible one for odd k's.

If k=2p, and $A^* = A$, then $A^2 \geqq 0$, so that by the Theorem

3.1

$$|A^k|_\infty = |(A^2)^p|_\infty = (1/2+\sqrt{n}/2)|A^2|_\sigma^p = (1/2+\sqrt{n}/2)|A|_\sigma^k .$$

Thus for even k's the bound (4) holds true; it follows from the theorem 3.1 that also this bound is the best possible one.

Since clearly $|A^k|_\infty^{1/k} \geq |A|_\sigma$ for each $k>0$, our results may be considered as the quantitative refinement of the spectral radius formula for the $l_\infty$-norm and symmetric matrices. We believe that they are of some practical interest.

I wish to thank Professor Vlastimil Pták and Vladimír Müller for reading the manuscript and suggesting some improvements.

R e f e r e n c e s

[1] Z. DOSTÁL: $l_\infty$ norm of iterates and the spectral radius of matrices, to be published.

[2] Z. DOSTÁL: Polynomials of the eigenvalues and powers of matrices, Comment. Math. Univ. Carolinae 19(1978), 459-469.

[3] J. MAŘÍK, V. PTÁK: Norms, spectra and combinatorial properties of matrices, Czech. Math. J. 85(1960), 181-196.

[4] V. PTÁK, N.J. YOUNG: Functions of operators and the spectral radius, to appear in Linear Algebra and its Appl.

Hornický ústav ČSAV
Hladnovská 7, 71000 Ostrava 2
Československo

(Oblatum 20.12.1979)