Commentationes Mathematicae Universitatis Carolinae

Bruce Glastad; Glenn Hopkins

Commutative semigroup rings which are principal ideal rings

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 371--377

Persistent URL: http://dml.cz/dmlcz/106003

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

21,2 (1980)

COMMUTATIVE SEMIGROUP RINGS WHICH ARE PRINCIPAL IDEAL RINGS

Bruce GLASTAD, Glenn HOPKINS

Abstract: All zero dimensional principal ideal rings which are also semigroup rings are homomorphic images of semigroup rings where the semigroup is of a certain restricted type. We study semigroup rings over fields and, more generally, over special primary rings [cf. 5]; we determine necessary and sufficient conditions in order that these be principal when the semigroup is of this restricted type.

Key words: Semigroup ring, principal ideal ring.

Classification: 13C05

---

1. Introduction. Let R be a commutative ring with identity and let S be an abelian semigroup with zero. Structural properties of R[X;S], the semigroup ring of S over R, have been studied in several papers [6, 7, 8]. In particular, R. Gilmer has shown that R[X;S] is a principal ideal domain if and only if R is a field and either S is isomorphic to the additive group of integers or S is isomorphic to the additive semigroup of non-negative integers. In this paper we obtain a generalization of the well-known result (cf. Maschke's theorem) that if F is a field and G is a finite abelian group, then F[X;G] is a principal ideal ring if and only if either the characteristic of F is 0 or the characteristic

of F is p and the p-primary subgroup of G is cyclic.

To this end, we begin by observing that since R is a homomorphic image of R[X;S], a necessary condition for R[X;S] to be principal is that R be principal, and so $R = \bigoplus_{i=1}^{n} R_i$, a finite direct sum of special primary rings and principal ideal domains. It follows that $R[X;S] = \bigoplus_{i=1}^{n} R_i[X;S]$. Hence R[X;S] is principal if and only if $R_i[X;S]$ is principal for each i, $1 \leq i \leq n$. Since R[X;S] is Noetherian, S is finitely generated [2]. This fact and the fact that the (Krull) dimension of a principal ring is at most 1, taken together with a result of Gilmer [8, Theorem 7] imply that when R is a special primary ring (which includes the field case), R[X;S] is zero dimensional and Noetherian if and only if S is finite.

2. Conditions under which R[X;S] is principal. Let S be a finite semigroup. Then S is a homomorphic image of a semigroup of the form $S_0 = \langle \{a_1,\ldots,a_n\} \mid e_i a_i = (e_i + f_i)a_i, i = 1,\ldots,n\rangle$ where $\{a_1,\ldots,a_n\}$ is a generating set for S_0 over Z_0, the non-negative integers, $\{e_1,\ldots,e_n\} \subseteq Z_0$, and $\{f_1,\ldots,f_n\} \subseteq N$, the positive integers. We shall assume that the f_i are minimal, so that the period [3, Section 1.6] of a_i is f_i and no other relations hold among the a_i. Recall that $R[X;S_0] \cong R[X_1,\ldots,X_n]/(X_1^{e_1}(1 - X_1^{f_1}),\ldots,X_n^{e_n}(1 - X_n^{f_n}))$, where $\{X_1,\ldots,X_n\}$ is a set of indeterminates over R; see [12]. We use this fact in the proof below.

Theorem 1. Let F be a field and let S_0 be as above. Then:

i) If char $F = 0$, then $F[X;S_0]$ is a principal ideal ring if and only if at most one of $e_1,\dots,e_n$ is greater than 1.

ii) If char $F = p$, then $F[X;S_0]$ is a principal ideal ring if and only if either each of $e_1,\dots,e_n$ is less than or equal to 1 and p is relatively prime to all but one of $f_1,\dots,f_n$ or exactly one of $e_1,\dots,e_n$ is greater than 1 - say e_1 - and p is relatively prime to $f_2,\dots,f_n$.

Proof. It is easily verified that if two or more of $e_1,\dots,e_n$ are greater than 1 then, for any field F, $F[X;S_0]$ has $F[X_i,X_j]/(X_i^2,X_j^2)$, some $i \neq j$, as a homomorphic image; since this ring is not principal a necessary condition for $F[X;S_0]$ to be principal is that at most one of $e_1,\dots,e_n$ be greater than 1.

When char $F = p$, and p divides two or more of $f_1,\dots,f_n$ say p divides f_1 and p divides f_2, we have a sequence of surjective maps:

$$\frac{F[X_1,\dots,X_n]}{(X_1^{e_1}(1 - X_1^{f_1}),\ X_2^{e_2}(1 - X_2^{f_2}),\ X_3^{e_3}(1 - X_3^{f_3}),\dots,X_n^{e_n}(1 - X_n^{f_n}))} \to$$

$$\frac{F[X_1,\dots,X_n]}{(X_1^{e_1}(1 - X_1^{f_1}),\ X_2^{e_2}(1 - X_2^{f_2}),\ (1 - X_3),\dots,(1 - X_n))} \to$$

$$\frac{F[X_1,X_2]}{(X_1^{e_1}(1 - X_1^{f_1}),\ X_2^{e_2}(1 - X_2^{f_2}))} \to \frac{F[X_1,X_2]}{((1 - X_1)^p,\ (1 - X_2)^p)} .$$

This last ring is not principal, for consider $M = (\bar{X}_1 - \bar{1}, \bar{X}_2 - \bar{1})$, a maximal ideal. Let $A = ((\bar{X}_1 - \bar{1})^2, \bar{X}_2 - \bar{1}, (\bar{X}_1 - \bar{1})(\bar{X}_2 - \bar{1}))$. Then we have the chain of proper containments $M \supset A \supset M^2$, which implies that M is not principal; see [5, p. 462]. Hence when char F = p, a necessary condition for $F[X;S_0]$ to be principal is that no more than one of $f_1,\ldots,f_n$ be divisible by p.

Conversely, suppose that $e_2,\ldots,e_n$ are less than or equal to 1. We have

$$\frac{F[X_1,\ldots,X_n]}{(X_1^{e_1}(1 - X_1^{f_1}),\ldots,X_n^{e_n}(1 - X_n^{f_n}))} \simeq \frac{F[X_1]}{(X_1^{e_1}(1 - X_1^{f_1}))} \otimes_F$$

$$\left(\frac{F[X_2]}{(X_2^{e_2}(1 - X_2^{f_2}))} \otimes_F \cdots \otimes_F \frac{F[X_n]}{(X_n^{e_n}(1 - X_n^{f_n}))}\right).$$

For each i greater than or equal to two,

$$\left(\frac{F[X_i]}{(X_i^{e_i}(1 - X_i^{f_i}))} \simeq \frac{F[X_i]}{(1 - X^{f_i})} \text{ (if } e_i = 0) \text{ or}\right.$$

$$\frac{F[X_i]}{(X_i^{e_i}(1 - X_i^{f_i}))} \simeq F \otimes \frac{F[X_i]}{(1 - X_i^{f_i})} \quad \text{(if } e_i = 1).$$

Now $\frac{F[X_i]}{(1 - X_i^{f_i})}$ is a finite direct sum of finite dimensional field extensions of F if char F = 0; if char F = p, the same is true if p does not divide f_i [9, p. 26].

If p divides f_i, some $i \geq 2$, and $e_1 > 1$, then in a manner similar to that used above, we obtain the non-principal ring $\frac{f[X_1,X_i]}{(X_1^{e_1},(1 - X_i)^p)}$ as a homomorphic image of $F[X;S_0]$. Hence for $F[X;S_0]$ to be principal in the characteristic p case, it is necessary that either: i) each of $e_1,\ldots,e_n$ is less than or equal to 1 and p divides at most one of $f_1,\ldots$ $\ldots,f_n$; or ii) exactly one of $e_1,\ldots,e_n$ is greater than 1 - say e_1 - and p is relatively prime to $f_2,\ldots,f_n$.

For arbitrary characteristic we have

$$\frac{F[X_1,\ldots,X_n]}{(X_1^{e_1}(1 - X_1^{f_1}),\ldots,X_n^{e_n}(1 - X_n^{f_n}))} \simeq$$

$$\frac{F[X_1]}{(X_1^{e_1}(1 - X_1^{f_1}))} \otimes_F \left(\frac{F[X_2]}{(X_2^{e_2}(1 - X_2^{f_2}))} \otimes_F \cdots \otimes_F \frac{F[X_n]}{(X_n^{e_n}(1 - X_n^{f_n}))}\right) =$$

$$\frac{F[X_1]}{(X_1^{e_1}(1 - X_1^{f_1}))} \otimes_F T ,$$

and, without loss of generality, in the characteristic p case, p is relatively prime to $f_2,\ldots,f_n$. Hence, for $i \geq 2$, $1 - X_i^{f_i}$ is a separable polynomial over F and so T is a finite direct sum of finite dimensional separable field extensions of F [1, pp. 236-238], say $T = \bigoplus_{i=1}^{q} K_i$. Therefore the ring is isomorphic to $\bigoplus_{i=1}^{q} \frac{K_i[X_1]}{(X_e^{e_1}(1 - X_1^{f_1}))}$ which is principal.

<u>Corollary</u>. Let A be a special primary ring which is not a field. If char A = 0, then $A[X;S_0]$ is principal if and only if each of $e_1,\ldots,e_n$ is less than or equal to 1; if char A = = p, then $A[X;S_0]$ is principal if and only if each of $e_1,\ldots$ $\ldots,e_n$ is less than or equal to 1 and each of $f_1,\ldots,f_n$ is relatively prime to p.

<u>Proof</u>. Using Cohen's Structure Theorem,

$$A[X;S_0] \simeq A \otimes_F F[X;S_0] \simeq \frac{F[X]}{(X^t)} \otimes_F F[X;S_0]$$

where F is a coefficient field of A; see [11, Section 31]. From the above we see that if char F = 0, then $A[X;S_0]$ is principal if and only if each of $e_1,\ldots,e_n$ is less than or equal to 1 (since t, the index of nilpotence of A, is greater than 1). The proof for char A = p is similar.

We note that the above theorem gives sufficient conditions for $F[X;S]$ to be principal when S is an arbitrary finite abelian semigroup with 0 (since $F[X;S]$ is a homomorphic image of $F[X;S_0]$). We also remark that in terms of the semigroup S_0 the theorem says that if char F = 0, then $F[X;S_0]$ is principal if and only if all but at most one of the cyclic subsemigroups $\langle a_i \rangle = \{a_i, 2a_i, \ldots\}$ is actually a subgroup of S_0 (not necessarily with the same identity as S_0); and if char F = p, then $F[X;S_0]$ is principal if and only if either all the $\langle a_i \rangle$ are subgroups and at most one of these subgroups has period divisible by p, or exactly one of the $\langle a_i \rangle$ - say $\langle a_1 \rangle$ - is not a subgroup and all other $\langle a_j \rangle$, $j \geq 2$, have period relatively prime to p.

We conclude by remarking that the techniques employed above can be used to study the structure of $A[X;S_0]$ when A is a special primary ring that does not contain a field. In this case, $A[X;S_0] \simeq A \otimes_W \Omega$, where W is a complete rank one discrete valuation ring of characteristic 0 and Ω is a tensor produce over W of rings of the form $\dfrac{W[X]}{(X^e(1 - X^f))}$.

R e f e r e n c e s

[1] Iain T. ADAMSON: Rings, Modules, and Algebras, Hafner Publishing Company, New York, 1971.

[2] L. BUDACH: Struktur Noetherscher kommutativer Halbgruppen, Monatsb. Deutsch. Akad. Wiss., Berlin 6(1964), 85-88.

[3] A.H. CLIFFORD and G.B. PRESTON: The Algebraic Theory of Semigroups, American Mathematical Society, Providence, Rhode Island, 1961.

[4] James L. FISHER and Sudarshan K. SEHGAL: Principal ideal group rings, Communications in Algebra 4(1976), 319-325.

[5] R. GILMER: Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972.

[6] R. GILMER and T. PARKER: Divisibility properties of semigroup rings, Michigan Math. J. 21(1974), 65-86.

[7] R. GILMER and T. PARKER: Semigroup rings as Prufer rings, Duke Math. J. 41(1974), 219-230.

[8] R. GILMER and Mark L. TEPLY: Idempotents of commutative semigroup rings, Houston Journal of Math. 3(1977), 369-385.

[9] I.N. HERSTEIN: Noncommutative Rings, Math. Association of America, 1958.

[10] Hideyuki MATSUMURA: Commutative Algebra, W.A. Benjamin, Inc., New York, 1970.

[11] Masayoshi NAGATA: Local Rings, Interscience Publishers New York, 1962.

[12] Tom PARKER: The Semigroup Ring, Dissertation, Florida State University, 1973.

University of Mississippi
Mississippi 38677
U.S.A.

(Oblatum 4.9. 1979)