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# ON THE INDIVIDUAL ERGODIC THEOREM ON A LOGIC

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Abstract: The individual ergodic theorem on a logic is formulated and proved.

Key words: Logic, state, observable, ergodic homomorphism.

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Let (X,S,m,T) be a classical dynamical system. The well-known Birkhoff individual ergodic theorem states (in the case that T is ergodic and f integrable) that the time mean

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$$

is equal a.e. to the space mean (phase mean)

$$\frac{1}{m(X)} \int_X f\, dm.$$

(See e.g. [4]; for recent development see [5],[6].) In the paper we shall formulate and prove a variant of the theorem for logics (orthomodular lattices) which are adequate to the quantum meachanical systems. (See [7], some connections to ergodic theory have been studied in [1].)

The main idea of our proof is to represent the given homomorphism $\tau$ of a logic L by a Borel measurable transformation of R. (A similar method in another area of non-commutative probability theory has been used in [2].) Of course, not every homomorphism $\tau$ permits such a representation: in Proposition 1 we present a sufficient and necessary condition ($x$-measurability of $\tau$). Under this condition all considered observables map the Borel $\sigma$-algebra $B(R_1)$ into a fixed Boolean algebra $x(B(R_1))$ and we could work with Boolean algebras instead of logics. Of course, such a specification presents a new result as well. On the other hand, it would be interesting to explain the physical meaning of the $x$-measurability of the homomorphism $\tau$; we do not know any convenient interpretation.

Let L be a logic, that is, L is a $\sigma$-lattice with the first and the last elements 0 and 1, respectively, with an orthocomplementation $\perp : a \mapsto a^{\perp}$, $a, a^{\perp} \in L$, which satisfies (i) $(a^{\perp})^{\perp} = a$ for all $a \in L$; (ii) if $a < b$, then $b^{\perp} < a^{\perp}$; (iii) $a \vee a^{\perp} = 1$ for all $a \in L$; and the orthomodular law holds in L: if $a < b$, then $b = a \vee (b \wedge a^{\perp})$.

We say that two elements $a, b \in L$ are (i) orthogonal, and we write $a \perp b$, if $a < b^{\perp}$; (ii) compatible, and we write $a \leftrightarrow b$, if there are three mutually orthogonal elements $a_1$, $b_1$, $c$ such that $a = a_1 \vee c$, $b = b_1 \vee c$.

An observable is a map $x$ from $B(R_1)$ into L such that (i) $x(\emptyset) = 0$; (ii) if $E \cap F = \emptyset$, then $x(E) \perp x(F)$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$, $E_i \cap E_j = \emptyset$, $i \neq j$, $E_i \in B(R_1)$. If $f$ is a Borel function, then $f \circ x : E \mapsto x(f^{-1}(E))$, $E \in B(R_1)$, is an observable. The null observable is the observable $\sigma$ such that $\sigma(\{0\}) = 1$.

Two observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for any $E,F \in B(R_1)$.

For compatible observables there is a calculus [7, Theorem 6.17]. Therefore we may define, for example, the sum $x_1 + \ldots + x_n$ for the compatible observables $x_1,\ldots,x_n$.

A state is a map $m:L \longrightarrow \langle 0,1 \rangle$ such that (i) $m(1) = 1$; (ii) $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ if $a_i \perp a_j$, $i \neq j$. If x is an observable, then the mean value of x in a state m is the expression $m(x) = \int_{R_1} t \, dm_x(t)$ (if the integral exists), where $m_x(E) = m(x(E))$, $E \in B(R_1)$.

A homomorphism of a logic L is a map $\tau$ from L into L such that (i) $\tau(0) = 0$; (ii) $\tau(a^{\perp}) = (\tau(a))^{\perp}$ for all $a \in L$; (iii) $\tau(\bigvee_{i=1}^{\infty} a_i) = \bigvee_{i=1}^{\infty} \tau(a_i)$, $\{a_i\}_{i=1}^{\infty} \subset L$.

We say that a homomorphism $\tau$ of a logic L is ergodic in a state m (see [1]) if

(i) $m(\tau(a)) = m(a)$ for all $a \in L$;

(ii) if $\tau(a) = a$, then $m(a) \in \{0,1\}$.

A homomorphism $\tau : L \longrightarrow L$ is said to be x-measurable if $\tau(x(B(R_1))) \subset x(B(R_1))$.

We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of observables converges to the null observable $\sigma$ almost everywhere [m] (a.e. [m], see [3,2]) if

$$m(\limsup_n x_n(\langle -\varepsilon, \varepsilon \rangle^c)) = 0$$

for every $\varepsilon > 0$.

Now we can formulate the individual ergodic theorem on a logic.

<u>Theorem</u>. Let x be an observable, $\tau$ an x-measurable homomorphism of a logic L, ergodic in a state m. Let $m(x) = 0$. Then

(1) $$\frac{1}{n}\sum_{i=1}^{n-1} \tau^{i} \circ x \longrightarrow \sigma \quad \text{a.e. } [m].$$

Proof. Our Theorem will be proved by means of the next Propositions.

Proposition 1. Let $x$ be an observable. A homomorphism $\tau : L \longrightarrow L$ is $x$-measurable iff there is a Borel measurable transformation $T: R_1 \longrightarrow R_1$ such that

(2) $$\tau \circ x = T \circ x.$$

(That is, $x(T^{-1}(E)) = \tau(x(E))$ for any $E \in B(R_1)$.)

Proof. The sufficient condition is evident. Conversely, let $\tau$ be an $x$-measurable homomorphism. This implies that if $E \subset F$, $E, F \in B(R_1)$ and if there is $G' \in B(R_1)$ such that $\tau(x(E)) < x(G') < \tau(x(F))$, then there is $G \in B(R_1)$ such that $E \subset G \subset F$, $x(G) = x(G')$. Indeed, if we put $G = (G' \cap F) \cup E$, then this $G$ has the claimed property.

Now, let $r_1, r_2, \ldots$ be any distinct enumeration of the rational numbers in $R_1$. We claim to construct, by induction, the sets $E_1, E_2, \ldots$ from $B(R_1)$ such that

(a) $x(E_i) = \tau(x((-\infty, r_i)))$;

(b) $E_i \subset E_j$ if $r_i < r_j$;

(c) $\bigcap_{i=1}^{\infty} E_i = \emptyset$.

Let $E_1$ be any set in $B(R_1)$ such that $x(E_1) = \tau(x((-\infty, r_1)))$. Suppose $E_1, \ldots, E_n \in B(R_1)$ have been constructed such that (a) and (b) hold. We shall construct $E_{n+1}$ as follows. Let $(i_1, \ldots, i_n)$ be the permutation of $(1, \ldots, n)$ such that $r_{i_1} < \ldots < r_{i_n}$. Then exactly one of the following conditions holds:

(i) $r_{n+1} < r_{i_1}$;

(3) (ii) $r_{n+1} > r_{i_n}$;

(iii) there is unique $k \in \{1,\dots,n\}$ such that

$$r_{i_k} < r_{n+1} < r_{i_{k+1}}.$$

By the above observation we can select $E_{n+1}$ such that (i) $E_{n+1} \subset E_{i_1}$; (ii) $E_{n+1} \supset E_{i_n}$; (iii) $E_{i_k} \subset E_{n+1} \subset E_{i_{k+1}}$; according to (3). Then the system $\{E_1,\dots,E_{n+1}\}$ fulfils (a) and (b). Thus, by induction, it follows that there exists a sequence $\{E_i\}_{i=1}^{\infty}$ of sets in $B(R_1)$ with the properties (a) and (b). As

$$x(\bigcap_{i=1}^{\infty} E_i) = \bigwedge_{i=1}^{\infty} x(E_i) = \bigwedge_{i=1}^{\infty} \tau(x((-\infty,r_i))) = 0,$$

we may, by replacing $E_i$ by $E_i - \bigcap_{j=1}^{\infty} E_j$ if necessary, assume that $\bigcap_{i=1}^{\infty} E_i = \emptyset$.

We define a $B(R_1)$-measurable transformation $T:R_1 \longrightarrow R_1$ as follows:

$$T(t) = \begin{cases} 0 & \text{if } t \notin \bigcup_{i=1}^{\infty} E_i \\ \inf\{r_j : t \in E_j\} & \text{if } t \in \bigcup_{i=1}^{\infty} E_i. \end{cases}$$

A transformation T is everywhere defined and it is finite. Moreover,

$$T^{-1}((-\infty,r_i)) = \begin{cases} \bigcup_{r_j < r_i} E_j & \text{if } r_i \leq 0 \\ \bigcup_{r_j < r_i} E_j \cup (R_1 - \bigcup_{k=1}^{\infty} E_k) & \text{if } r_i > 0. \end{cases}$$

Hence T is $B(R_1)$-measurable and $x(T^{-1}((-\infty,r_i))) = \tau(x((-\infty, r_i)))$. Therefore $x(T^{-1}(E)) = \tau(x(E))$ for any $E \in B(R_1)$ and the necessary condition is proved. Q.E.D.

<u>Proposition 2</u>. Let x be an observable. If a homomorph-

ism $\tau : L \longrightarrow L$ is x-measurable, then for the above transformation T we have

$$\tau^n \circ x = T^n \circ x, \quad n = 1,2,\dots .$$

If $\tau$ is an ergodic homomorphism in a state m, then T is an $m_x$-measure preservative ergodic transformation from $R_1$ into itself.

Proof. The first part is evident by induction.
Let $\tau$ be ergodic. Then, by Proposition 1, we have
$m_x(T^{-1}(E)) = m(x(T^{-1}(E))) = m(\tau(x(E))) = m(x(E)) = m_x(E)$, $E \in B(R_1)$.

Further, if $T^{-1}(E) = E$, then $x(T^{-1}(E)) = x(E)$, $\tau(x(E)) = x(E)$. Due to the ergodicity of $\tau$ we conclude that $m(x(E)) = m_x(E) \in \{0,1\}$. Q.E.D.

Proof of Theorem. From the assumption of Theorem we conclude that $\tau^n \circ x = T^n \circ x$, where T is an ergodic transformation with respect to the measure $m_x$ on $B(R_1)$, and the observables $\{\tau^n \circ x\}_{n=0}^{\infty}$ are mutually compatible. If we put $s_n = 1/n \sum_{i=0}^{n-1} T^i$, then, due to the calculus for compatible observables, the observables $y_n = s_n \circ x$ are the Cesàro sum $1/n \sum_{i=0}^{n-1} \tau^i \circ x$.

Since it may be shown that (see [3])

$$\frac{1}{n} \sum_{i=0}^{n-1} \tau^i \circ x \longrightarrow \sigma \quad \text{a.e. } [m] \text{ iff } s_n \longrightarrow 0 \text{ a.e. } [m_x] ,$$

we conclude, from the validity of the individual ergodic theorem on the dynamical system $(R_1, B(R), m_x, T)$ applied to the identical function $i(t) = t$, $t \in R_1$, ( $\int_{R_1} i(t)\, dm_x(t) = 0$) [4], that (1) holds. Q.E.D.

## R e f e r e n c e s

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