

David R. Jackett

The nil-degree of a torsion-free Abelian group

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 2, 393--406

Persistent URL: <http://dml.cz/dmlcz/106006>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1980

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

THE NIL-DEGREE OF A TORSION-FREE ABELIAN GROUP
D. R. JACKETT

Abstract: Recently Webb, Acta Sci. Math. Szeged 39 (1977), 185-188 provided a bound for the nil-degree (if it is finite) of a torsion free group of finite rank. In this paper we extend Webb's result to torsion-free groups A , not necessarily of finite rank, but with certain finiteness conditions on the rank of A/pA for each prime p . We also prove an associative ring on such a group is nilpotent exactly if it is nil.

Key words: Ring, (strong) nil-degree, p -adic module.

Classification: 20K20

All groups that we consider here are abelian groups, and all rings are not necessarily associative rings. A ring on a group A is a ring whose additive group is (isomorphic to) A . We write (A, \cdot) for a ring on A and say that A supports (A, \cdot) . The rank of A is denoted by $r(A)$. We use the standard notation \mathbb{Z} , and for a prime p , J_p for the group of integers and the group of p -adic integers, respectively.

Szele [9] defined the nil degree (nilstufe) of a group A to be ∞ or the largest integer n (if one exists) such that there is an associative ring (A, \cdot) on A with $(A, \cdot)^n \neq 0$. Gard-

This paper formed part of the author's Ph.D. thesis, University of Tasmania, 1977, which was written under the direction of Dr. B.J. Gardner.

ner [8] defined the strong nil-degree of A similarly, where for the non-associative ring (A, \cdot) on A , $(A, \cdot)^n$ is the subring of (A, \cdot) generated by all products of the form $(\dots((a_1 \cdot a_2) \cdot a_3) \cdot \dots) \cdot a_n$. Throughout this paper $(A, \cdot)^n$ will always have this meaning. Feigelstock [4] has introduced a concept very similar to the strong nil-degree of a torsion-free group. Following Feigelstock we define the extra strong nil-degree (strong nilstufe) of the torsion-free group A to be the positive integer n such that there is a ring on A with a non-zero product of length n (all possible bracketings considered), but no ring on A with non-zero products of length greater than n . If no such n exists then the extra strong nil-degree is defined to be ∞ . For a torsion-free group A we let $N(A)$, $N_S(A)$ and $N_E(A)$ respectively denote the nil-degree, the strong nil-degree and the extra strong nil-degree of A . A group is called nil if it has nil-degree 1.

Feigelstock [5] has claimed that if A is a torsion-free group of rank two then $N_E(A)$ is 1, 2, or ∞ , but appears to have only shown that $N(A)$ is 1, 2, or ∞ ; his proof relies on Lemma 1 of Beaumont and Wisner [3] that requires consideration of associative rings. Feigelstock has also conjectured that if A is a torsion-free group of finite rank n then $N_E(A)$ is 1, 2, ..., n or ∞ .

Recently Webb [10] has shown that if A is a torsion-free group of rank n then $N(A)$ is 1, 2, ..., n or ∞ and $N_E(A)$ is 1, 2, ..., 2^{n-1} or ∞ . Also, he has provided an example of a torsion-free group A of rank three for which $N_E(A) = 4$. Thus Feigelstock's conjecture is not true. However, if we replace $N_E(A)$ with $N_S(A)$, the conjecture can be proved.

Theorem 1. Let (A, \cdot) be a ring on a torsion-free group
 A of finite rank n . If $(A, \cdot)^m = 0$ for some positive integer
 m then $(A, \cdot)^{n+1} = 0$.

Proof: Suppose $(A, \cdot)^m = 0$ for some positive integer m ,
and k is a positive integer for which $(A, \cdot)^{k+1} \neq 0$. We show
 $(A, \cdot)^k / (A, \cdot)^{k+1}$ is not a torsion group.

Indeed, suppose $(A, \cdot)^k / (A, \cdot)^{k+1}$ is torsion. If we choose
a non-zero element $a \in (A, \cdot)^k$ then there is an integer $n_1 \neq 0$
such that $n_1 a \in (A, \cdot)^{k+1}$. Thus

$$0 \neq n_1 a = a'_1 \cdot a_1 + a'_{1_2} \cdot a_{1_2} + a'_{1_3} \cdot a_{1_3} + \dots + a'_{1_{n(1)}} \cdot a_{1_{n(1)}},$$

where a_1 and a_{1_i} are in A , and a'_1 and a'_{1_i} are in $(A, \cdot)^k$, for
each $i \in \{2, 3, \dots, n(1)\}$. Without loss of generality we can assume
 $a'_1 \cdot a_1 \neq 0$.

Since $a'_1 \in (A, \cdot)^k$ it is possible to choose a non-zero integer
 n_2 such that $n_2 a'_1 \in (A, \cdot)^{k+1}$. Hence

$$0 \neq n_2 (a'_1 \cdot a_1) = (a'_2 \cdot a_2 + a'_{2_2} \cdot a_{2_2} + a'_{2_3} \cdot a_{2_3} + \dots +$$

$$+ a'_{2_{n(2)}} \cdot a_{2_{n(2)}}) \cdot a_1,$$

where a_2 and a_{2_i} are in A , and a'_2 and a'_{2_i} are in $(A, \cdot)^k$, for
each $i \in \{2, 3, \dots, n(2)\}$. Again we can assume $(a'_2 \cdot a_2) \cdot a_1 \neq 0$.

If we repeat this procedure we can obtain elements $a_1,$
 a_2, \dots, a_{m-k} in A , and an element a'_{m-k} in $(A, \cdot)^k$ such that

$$(\dots((a'_{m-k} \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \dots) \cdot a_1 \neq 0.$$

Clearly

$$(\dots((a'_{m-k} \cdot a_{m-k}) \cdot a_{m-k-1}) \cdot \dots) \cdot a_1 \in (A, \cdot)^m$$

contradicting the fact that $(A, \cdot)^m = 0$. We conclude that

$(A, \cdot)^k / (A, \cdot)^{k+1}$ cannot be a torsion group.

Consequently, for each positive integer k for which $(A, \cdot)^{k+1} \neq 0$, $(A, \cdot)^k / (A, \cdot)^{k+1}$ has torsion-free rank greater than zero. That is, $r((A, \cdot)^k)$ is strictly greater than $r((A, \cdot)^{k+1})$. Since A has finite rank n , $(A, \cdot)^{n+1} = 0$.

Corollary 2. If A is a torsion-free group of rank n then $N_S(A)$ is $1, 2, \dots, n$ or ∞ .

It is not difficult to find torsion-free groups A of rank n for which the bound of n for $N_S(A)$ in Corollary 2 is actually attained. Indeed consider $A = \bigoplus_{i=1}^n A_i$ where each A_i is a rational group with type $(2i, 2i, \dots, 2i, \dots)$. A reference to Theorem 4.2 of Gardner [8] shows $N_S(A) = n$.

The remainder of this paper is concerned with extending the associative case of Corollary 2 (that is, Webb's Theorem) to other classes of torsion-free groups. Our aim is two-fold: we wish to find some infinite rank torsion-free groups whose nil degrees, if finite, are bounded, and we would also like under certain circumstances to lower the bound on the finite nil-degrees mentioned in the Corollary. We concentrate our attention on torsion-free groups A with the property that for each prime p , $r(A/pA)$ is bounded by some positive integer n (not depending on p). This amounts to considering torsion-free groups whose p -basic subgroups all have rank $\leq n$. Clearly a torsion-free group of rank n satisfies this property.

For a group A and a prime p let $\hat{A}_{(p)} = \varprojlim_{\mathcal{N}} (A/p^k A)$ denote the p -adic completion of A . If A is torsion-free and p -reduced then clearly $\hat{A}_{(p)}$ is torsion-free. Also, $\hat{A}_{(p)}$ can

be made into a module over the ring of p -adic integers \mathbb{Q}_p^* by defining, for $j = s_0 + s_1p + \dots + s_kp^k + \dots$ in \mathbb{Q}_p^* and $(a_1 + pA, a_2 + p^2A, \dots, a_k + p^kA, \dots)$ in $\hat{A}_{(p)}$,

$$\begin{aligned} & j(a_1 + pA, a_2 + p^2A, \dots, a_k + p^kA, \dots) \\ &= (j^{(1)}(a_1 + pA), j^{(2)}(a_2 + p^2A), \dots, j^{(k)}(a_k + p^kA), \dots) \end{aligned}$$

where $j^{(k)} = s_0 + s_1p + \dots + s_{k-1}p^{k-1}$ for each positive integer k .

The next result enables us to extend rings on certain groups to rings on their p -adic completions.

Proposition 3. Suppose A is a group with no elements of infinite p -height for some prime p , and (A, \cdot) is a ring on A . Then there is exactly one ring structure $(\hat{A}_{(p)}, \cdot)$ on $\hat{A}_{(p)}$ which extends that of (A, \cdot) , and this preserves associativity and commutativity in (A, \cdot) .

Furthermore $(\hat{A}_{(p)}, \cdot)$ becomes a \mathbb{Q}_p^* -algebra.

Proof: The proof of the Proposition is analogous to the proof of Corollary 119.4 of Fuchs [7]. The only statements that require verification are that the extension $(\hat{A}_{(p)}, \cdot)$ of (A, \cdot) is unique, and that $(\hat{A}_{(p)}, \cdot)$ becomes a \mathbb{Q}_p^* -algebra. Since A can be regarded as a p -pure and p -dense subgroup of the p -reduced group $\hat{A}_{(p)}$ the proof of Lemma 119.2 of Fuchs [7] applies to show that $(\hat{A}_{(p)}, \cdot)$ is unique. That $(\hat{A}_{(p)}, \cdot)$ is a \mathbb{Q}_p^* -algebra follows at once from the definition of the \mathbb{Q}_p^* -module $\hat{A}_{(p)}$ given prior to the Proposition.

The following well known result is required.

(4) (Fuchs [6], p. 166). Let $0 \rightarrow B \xrightarrow{\alpha} A \xrightarrow{\beta} C \rightarrow 0$ be a p-pure exact sequence. Then the sequence

$$0 \rightarrow \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \xrightarrow{\hat{\beta}} \hat{C}_{(p)} \rightarrow 0$$

is splitting exact.

Lemma 5. Suppose A is a torsion-free group and B is a p-basic subgroup of A. Then $\hat{A}_{(p)}$ and $\hat{B}_{(p)}$ are isomorphic p-adic modules. Furthermore, $\hat{A}_{(p)}$ has finite rank over Q_p^* if and only if B has finite rank over Z, and in this case the Q_p^* -rank of $\hat{A}_{(p)}$ and the Z-rank of B coincide.

Proof: Consider the p-pure exact sequence

$$0 \rightarrow B \xrightarrow{\alpha} A \rightarrow A/B \rightarrow 0$$

where α is the inclusion map. (4) shows that the sequence

$$0 \rightarrow \hat{B}_{(p)} \xrightarrow{\hat{\alpha}} \hat{A}_{(p)} \rightarrow (A/B)_{(p)} \rightarrow 0$$

is splitting exact, so $\hat{A}_{(p)} \cong \text{Im } \hat{\alpha} \oplus (A/B)_{(p)}$. Since A/B is p-divisible, $(A/B)_{(p)} = 0$, whence $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ (as groups).

Next let $(b_1 + pB, b_2 + p^2B, \dots, b_k + p^k B, \dots)$ be an arbitrary element of $\hat{B}_{(p)}$, and let j be a p-adic integer. Then

$$\begin{aligned} & \hat{\alpha}(j(b_1 + pB, b_2 + p^2B, \dots, b_k + p^k B, \dots)) \\ &= \hat{\alpha}(j^{(1)}b_1 + pB, j^{(2)}b_2 + p^2B, \dots, j^{(k)}b_k + p^k B, \dots) \\ &= (j^{(1)}b_1 + pA, j^{(2)}b_2 + p^2A, \dots, j^{(k)}b_k + p^k A, \dots) \\ &= j(b_1 + pA, b_2 + p^2A, \dots, b_k + p^k A, \dots) \\ &= j(\hat{\alpha}(b_1 + pB, b_2 + p^2B, \dots, b_k + p^k B, \dots)), \end{aligned}$$

so $\hat{A}_{(p)}$ and $\hat{B}_{(p)}$ are isomorphic Q_p^* -modules.

Suppose now the rank of B is finite. A trivial induc-

tion argument together with (4) show that the rank of $\hat{B}_{(p)}$ over Q_p^* is precisely the rank of B. Thus the Q_p^* -rank of $\hat{A}_{(p)}$ is the rank of B. To prove the converse suppose $\hat{A}_{(p)}$ has finite rank n over Q_p^* , and B has rank strictly greater than n. Then B contains a p-pure free summand of rank (n+1), so (4) shows that $\hat{A}_{(p)} \cong \hat{B}_{(p)}$ contains a summand isomorphic to the direct sum of (n+1) copies of J_p . This is clearly impossible.

Suppose A is a torsion-free group and $\alpha : A \rightarrow \hat{A}_{(p)}$ is the canonical map from A into its p-adic completion. If a is an arbitrary element of A then let \hat{a} denote the image of a under the map α . Similarly if B is a p-basic subgroup of A and $\beta : B \rightarrow \hat{B}_{(p)}$ is the canonical map from B into its p-adic completion, then let \bar{b} denote the image of $b \in B$ under the map β . We can now improve the final assertion in Lemma 5.

Lemma 6. Let A be a torsion-free group with finite rank p-basic subgroup $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \dots \oplus \langle b_n \rangle$. Then the elements $\hat{b}_1, \hat{b}_2, \dots, \hat{b}_n$ of $\hat{A}_{(p)}$ form a basis of $\hat{A}_{(p)}$ over Q_p^* .

Proof: From Lemma 5 it suffices to show that the set $S = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\}$ of elements of $\hat{B}_{(p)}$ form a basis of $\hat{B}_{(p)}$ over Q_p^* .

First we show that S is independent over Q_p^* . Indeed suppose

$$(*) \quad j_1 \bar{b}_1 + j_2 \bar{b}_2 + \dots + j_n \bar{b}_n = \bar{0}$$

for some p-adic integers j_1, j_2, \dots, j_n . With $j_i^{(k)}$ defined as usual for $i \in \{1, 2, \dots, n\}$ and $k \in \{1, 2, \dots\}$, (*) becomes

$$\begin{aligned}
& (j_1^{(1)}b_1 + pB, j_1^{(2)}b_1 + p^2B, \dots, j_1^{(k)}b_1 + p^kB, \dots) + \\
& + (j_2^{(1)}b_2 + pB, j_2^{(2)}b_2 + p^2B, \dots, j_2^{(k)}b_2 + p^kB, \dots) + \dots + \\
& + (j_n^{(1)}b_n + pB, j_n^{(2)}b_n + p^2B, \dots, j_n^{(k)}b_n + p^kB, \dots) \\
& = (pB, p^2B, \dots, p^kB, \dots).
\end{aligned}$$

Thus

$$j_1^{(k)}b_1 + j_2^{(k)}b_2 + \dots + j_n^{(k)}b_n \in p^kB$$

for each $k \in \{1, 2, \dots\}$. Hence for every $k \in \{1, 2, \dots\}$ there are integers $\ell_1^{(k)}, \ell_2^{(k)}, \dots, \ell_n^{(k)}$ such that

$$\begin{aligned}
j_1^{(k)}b_1 + j_2^{(k)}b_2 + \dots + j_n^{(k)}b_n &= \ell_1^{(k)}p^kb_1 + \ell_2^{(k)}p^kb_2 + \dots + \\
&+ \ell_n^{(k)}p^kb_n.
\end{aligned}$$

Consequently $j_i^{(k)} = \ell_i^{(k)}p^k$ for each $i \in \{1, 2, \dots, n\}$. But then

$$\begin{aligned}
j_i \bar{b}_i &= (j_i^{(1)}b_i + pB, j_i^{(2)}b_i + p^2B, \dots, j_i^{(k)}b_i + p^kB, \dots) \\
&= (\ell_i^{(1)}pb_i + pB, \ell_i^{(2)}p^2b_i + p^2B, \dots, \ell_i^{(k)}p^kb_i + p^kB, \dots) \\
&= \bar{0},
\end{aligned}$$

for each $i \in \{1, 2, \dots, n\}$. Since $\hat{B}_{(p)}$ is torsion-free as a Q_p^* -module, we conclude that S is independent over Q_p^* .

Next we show that S generates $\hat{B}_{(p)}$. Let

$$(b^{(1)} + pB, b^{(2)} + p^2B, \dots, b^{(k)} + p^kB, \dots)$$

be an arbitrary element of $\hat{B}_{(p)}$. Then for each $k \in \{1, 2, \dots\}$ there are suitable integers $m_i^{(k)}, i \in \{1, 2, \dots, n\}$, such that

$$b^{(k)} + p^kB = (m_1^{(k)}b_1 + m_2^{(k)}b_2 + \dots + m_n^{(k)}b_n + p^kB,$$

and $0 \leq m_i^{(k)} < p^k$. Now for each $k \in \{1, 2, \dots\}$

$$b^{(k+1)} + p^k_B = b^{(k)} + p^k_B,$$

so $b^{(k+1)} - b^{(k)} \in p^k_B$. It follows that for each $i \in \{1, 2, \dots, n\}$ and each $k \in \{1, 2, \dots\}$, $(m_i^{(k+1)} - m_i^{(k)})b_i \in p^k < b_i >$. Thus for each $i \in \{1, 2, \dots, n\}$, the sequence $m_i^{(1)}, m_i^{(2)}, \dots, m_i^{(k)}, \dots$ has the property that $m_i^{(k+1)} \equiv m_i^{(k)} \pmod{p^k}$, for each $k \in \{1, 2, \dots\}$. Hence $m_i^{(1)}, m_i^{(2)}, \dots, m_i^{(k)}, \dots$ determines a p -adic integer j_i for which $j_i^{(k)} = m_i^{(k)}$ for each $k \in \{1, 2, \dots\}$. But then

$$\begin{aligned} & j_1 \bar{b}_1 + j_2 \bar{b}_2 + \dots + j_n \bar{b}_n \\ &= (m_1^{(1)} b_1 + p_B, m_1^{(2)} b_1 + p^2_B, \dots, m_1^{(k)} b_1 + p^k_B, \dots) + \\ &+ (m_2^{(1)} b_2 + p_B, m_2^{(2)} b_2 + p^2_B, \dots, m_2^{(k)} b_2 + p^k_B, \dots) + \dots + \\ &+ (m_n^{(1)} b_n + p_B, m_n^{(2)} b_n + p^2_B, \dots, m_n^{(k)} b_n + p^k_B, \dots) \\ &= ((m_1^{(1)} b_1 + m_2^{(1)} b_2 + \dots + m_n^{(1)} b_n) + p_B, \\ &\quad (m_1^{(2)} b_1 + m_2^{(2)} b_2 + \dots + m_n^{(2)} b_n) + p^2_B, \dots \\ &\quad \dots, (m_1^{(k)} b_1 + m_2^{(k)} b_2 + \dots + m_n^{(k)} b_n) + p^k_B, \dots) \\ &= (b^{(1)} + p_B, b^{(2)} + p^2_B, \dots, b^{(k)} + p^k_B, \dots), \end{aligned}$$

so S indeed generates $\hat{B}_{(p)}$.

A consequence of Lemma 6 and Proposition 3 is the following.

Proposition 7. Suppose A is a torsion-free group with no elements of infinite p -height for some prime p , and $r(A/pA)$ is finite. Then any ring (A, \cdot) on A is completely determined by its effect upon any p -basic subgroup of A .

If A has finite rank and $r(A) = r(A/pA)$, then it is possible to choose a p-basic subgroup of A that is also a subring of (A, \cdot) .

Proof: Let $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \dots \oplus \langle b_n \rangle$ be a p-basic subgroup of A. If $\hat{A}_{(p)}$ is the p-adic completion of A then Proposition 3 shows that (A, \cdot) may be viewed as a subring of $(\hat{A}_{(p)}, \cdot)$. Lemma 6 now shows that the ring $(\hat{A}_{(p)}, \cdot)$, and hence the ring (A, \cdot) , is determined by the effect of (A, \cdot) on the set $\{b_1, b_2, \dots, b_n\}$.

To prove the final assertion of the Proposition we use an argument similar to the proof of Lemma 4.3 of Beaumont and Pierce [2]. Suppose $r(A) = r(A/pA) = n$. Then $\{b_1, b_2, \dots, b_n\}$ is a maximal independent set of elements of A, so for all i and $j \in \{1, 2, \dots, n\}$ there exists an integer m with $(m, p) = 1$, and integers m_1, m_2, \dots, m_n such that

$$m(b_i \cdot b_j) = m_1 b_1 + m_2 b_2 + \dots + m_n b_n .$$

Consequently $(mB, \cdot) = (\langle mb_1, mb_2, \dots, mb_n \rangle, \cdot)$ is a subring of (A, \cdot) . Finally since B is p-pure in A and $(m, p) = 1$ it follows that mB is a p-basic subgroup of A.

The partial similarity of Proposition 7 with Theorem 120.1 of Fuchs [7] cannot be strengthened. To demonstrate this simply let A be a rational group with non-idempotent type. It is clear that there is a prime p for which A satisfies the conditions of Proposition 7. However since A is a nil group and every p-basic subgroup of A is cyclic, not every partial multiplication on a p-basic subgroup of A will extend to a ring on A.

Suppose now A is a torsion-free group with no elements of infinite p -height, for some prime p , and (A, \cdot) is an associative ring on A . Proposition 3 shows that (A, \cdot) can be viewed as a subring of an associative ring $(\hat{A}_{(p)}, \cdot)$ on $\hat{A}_{(p)}$: If we let K denote the quotient field of \mathbb{Q}_p^* , then $K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}$ can be made into an associative algebra $(K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}, \cdot)$ over K by defining, for k_1, k_2 in K and \hat{a}_1, \hat{a}_2 in $\hat{A}_{(p)}$,

$$(k_1 \otimes \hat{a}_1) \cdot (k_2 \otimes \hat{a}_2) = (k_1 k_2) \otimes (\hat{a}_1 \cdot \hat{a}_2)$$

and

$$k_1(k_2 \otimes \hat{a}_1) = (k_1 k_2) \otimes \hat{a}_1.$$

It is clear that if $\hat{A}_{(p)}$ has finite rank over \mathbb{Q}_p^* then $K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}$ will have finite dimension over K . Also the map $\hat{a} \rightarrow 1 \otimes \hat{a}$ for each $\hat{a} \in \hat{A}_{(p)}$ is an embedding of $(\hat{A}_{(p)}, \cdot)$ in $(K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}, \cdot)$, so (A, \cdot) can be viewed as a subring of the algebra $(K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}, \cdot)$.

These comments form the basis for the proof of our next result.

Proposition 8. Let A be a torsion-free group with no elements of infinite p -height for some prime p , and suppose $r(A/pA) = n$. If (A, \cdot) is a nil ring then $(A, \cdot)^{n+1} = 0$.

Proof: (A, \cdot) can be embedded in the associative algebra $(K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}, \cdot)$ over the field K . If B is a p -basic subgroup of A then there exist elements b_1, b_2, \dots, b_n of A such that $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus \dots \oplus \langle b_n \rangle$. Lemma 6 shows that $\{\hat{b}_1, \hat{b}_2, \dots, \hat{b}_n\}$ is now a basis of $\hat{A}_{(p)}$ over \mathbb{Q}_p^* , so $\{1 \otimes \hat{b}_1, 1 \otimes \hat{b}_2, \dots, 1 \otimes \hat{b}_n\}$ is a basis of $K \otimes_{\mathbb{Q}_p^*} \hat{A}_{(p)}$ over K .

For each $i \in \{1, 2, \dots, n\}$, b_i is a nilpotent element of (A, \cdot) , so $1 \otimes \hat{b}_i$ is a nilpotent element of

$(K \otimes_{\mathbb{Q}_p} \hat{A}_{(p)}, \cdot)$. Since $(K \otimes_{\mathbb{Q}_p} \hat{A}_{(p)}, \cdot)$ has finite dimension n over K , a reference to Abian [1], p. 155, now shows $(K \otimes_{\mathbb{Q}_p} \hat{A}_{(p)}, \cdot)^{n+1} = 0$. Thus $(A, \cdot)^{n+1} = 0$, as desired.

Now for the main results.

Theorem 9. Suppose $A = D \oplus R$ is a torsion-free group, where D is a divisible group and R is a reduced group. Suppose further that D has finite rank d and the rank of A/pA is bounded by the integer n , for every prime p . If (A, \cdot) is a nil ring on A then $(A, \cdot)^{(d+1)(n+1)} = 0$.

Proof: Let (A, \cdot) be a nil ring on A . If there is a prime p for which A has no elements of infinite p -height, then Proposition 8 shows $(A, \cdot)^{n+1} = 0$. Hence we can assume that A has elements of infinite p -height for every prime p .

Consider a fixed prime p . It is readily checked that $A/p^\omega A$ is a torsion-free group with no elements of infinite p -height such that $r((A/p^\omega A)/p(A/p^\omega A)) \leq n$. Also, since $p^\omega A$ is a fully invariant subgroup of A , the nil ring (A, \cdot) on A yields a nil ring $(A/p^\omega A, \cdot)$ on $A/p^\omega A$. Thus Proposition 8 implies $(A/p^\omega A, \cdot)^{n+1} = 0$. Since this is true for every prime p , $(A, \cdot)^{n+1} \subseteq \bigcap_p p^\omega A = D$.

Now (D, \cdot) is an ideal of (A, \cdot) , so (D, \cdot) is also a nil ring. Since (D, \cdot) can be made into a finite dimensional algebra over the field of rationals \mathbb{Q} , (D, \cdot) is a nilpotent ring. Theorem 1 now shows $(D, \cdot)^{d+1} = 0$, so $(A, \cdot)^{(n+1)(d+1)} = 0$, as required.

Corollary 10. Let A be a reduced torsion-free group with the property that $r(A/pA)$ is bounded by the positive integer n, for every prime p. Then $N(A)$ is $1, 2, \dots, n$ or ∞ .

We conclude by noting that certain results in Webb [10] enable us to give the non-associative analogues of the previous Theorem and its Corollary. The proofs are omitted since they are direct consequences of the non-associative results in Webb's work and the arguments used to prove Theorem 9.

Theorem 11. Let $A = D \oplus R$ be a torsion-free group where D is a divisible group and R is a reduced group. Suppose D has finite rank d and the rank of A/pA is bounded by the integer n, for every prime p. If (A, \cdot) is a ring on A for which there is a positive integer m such that every product of length m is zero, then every product of length $(2^{n-1} + 1)(2^{d-1} + 1)$ is zero.

Corollary 12. Suppose A is a reduced torsion-free group with the property that $r(A/pA)$ is bounded by the positive integer n, for every prime p. Then $N_n(A)$ is $1, 2, \dots, 2^{n-1}$ or ∞ .

R e f e r e n c e s

- [1] A. ABIAN: Linear associative algebras, Pergamon Press, New York, 1971.
- [2] R.A. BEAUMONT and R.S. PIERCE: Torsion-free rings, Illinois J. Math. 5(1961), 61-98.
- [3] R.A. BEAUMONT and R.J. WISNER: Rings with additive group which is a torsion-free group of rank two, Acta Sci Math. Szeged 20(1959), 105-116.

- [4] S. FEIGELSTOCK: On the nilstufe of homogeneous groups, Acta Sci. Math. Szeged 36(1974), 27-28.
- [5] S. FEIGELSTOCK: The nilstufe of rank two torsion-free groups, Acta Sci. Math. Szeged 36(1974), 29-32.
- [6] L. FUCHS: Infinite abelian groups, Vol. I, Academic Press, New York, 1970.
- [7] L. FUCHS: Infinite abelian groups, Vol. II, Academic Press, New York, 1973.
- [8] B.J. GARDNER: Rings on completely decomposable torsion-free abelian groups, Comment. Math. Univ. Carolinae 15(1974), 381-392.
- [9] T. SZELE: Gruppentheoretische Beziehungen bei gewissen Ringkonstruktionen, Math. Z. 54(1951), 168-180.
- [10] M.C. WEBB: A bound for the nilstufe of a group, Acta Sci. Math. Szeged 39(1977), 185-188.

University of Tasmania
 Hobart
 Australia

(Oblatum 22.1. 1980)