Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 3, 447--456

Persistent URL: http://dml.cz/dmlcz/106011

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21,3 (1980)

REMARKS ON TOLERANCE SEMIGROUPS

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Abstract: The paper is devoted to a study of generalized fixed-points in finite tolerance semigroups.

Key words: Tolerance space, tolerance semigroup, connectedness, p-contractibility, generalized fixed-point.

Classification: 20M15

This investigation of tolerance semigroups has been motivated by a wish to find discrete versions to some known and deep theorems on topological semigroups. Our theorem 1 is analogous to a theorem of K.H.Hofmann and P.S.Mostert (see [2],p. 62,Theorem I) which reads as follows

Theorem (H.-M.) Let S be a compact connected semigroup with identity and $\mathcal{A}$ a compact connected abelian group of automorphisms of S. Then the set of fixed points of $\mathcal{A}$ on S is a compact connected subsemigroup which meets the minimal ideal.

Our theorem 2 resembles the Second fundamental theorem of compact semigroups in [2],p. 157. For a short history of tolerance structures and for bibliography, especially for that on tolerance algebras, see [1].

1. Basic definitions and notation. A tolerance t on a set X is any reflexive and symmetric binary relation on X. A set X together with a tolerance on it is called a tolerance space. The transitive closure $\bar{t}$ of a tolerance t on X is an equivalence relation on X. If $\bar{t}$ equals the universal relation on X, the

tolerance space X will be called <u>connected</u>.

Tolerances of different tolerance spaces will mostly be denoted by the same symbol t provided this will not give cause to any misunderstanding. Sometimes, if purposeful, X_C will denote the underlying set of the tolerance space X.

Having tolerance spaces X and Y, a <u>tolerance mapping</u> (or <u>continuous mapping</u>) $f:X\longrightarrow Y$ is any mapping $f:X_C\longrightarrow Y_C$ which preserves the tolerance relation. This means that for all a, $b\in X$, a t b in X implies f(a) t f(b) in Y.

The <u>cartesian product</u> $X\times Y$ is defined by $(X\times Y)_C = X_C\times Y_C$ and by the following convention: we set (a,c) t (b,d) in $X\times Y$ if and only if atb in X and ctd in Y.

The set T(X) of all tolerance mappings $X\longrightarrow X$ is made to a tolerance space mostly by taking the following tolerance p on T(X): for any $f,g\in T(X)$ we set f p g if and only if for all $a,b\in X$, a t b implies f(a) t g(b). If T(X) together with p is connected, X will be said to be <u>p-contractible</u>.

Let us have a tolerance space X with a tolerance t. A subset $A\subset X_C$ will be called a <u>simplex</u> in X if and only if $A\times A\subset t$. We have then a t b for all $a,b\in A$. If $f\in T(X)$ and $\{f(a)\mid a\in A\}=$ $= f(A) = A$, the simplex A will be said to be <u>fixed under</u> f. Let $F\subset T(X)$. Any $a\in X$ will be said to be a <u>generalized fixed-point of</u> F if and only if there is some simplex A in X fixed under all $f\in F$ with $a\in A$. (This treatment of fixed-points comes essentially from [3].)

A <u>tolerance semigroup</u> S is a compound notion: S is supposed to be a tolerance space and a semigroup. It is supposed that the semigroup operation $S\times S\longrightarrow S$ is a tolerance mapping. The last condition can be given the following form: if a t b and c t d for some $a,b,c,d\in S$, then ac t bd. An <u>automorphism</u>

of a tolerance semigroup S is an automorphism of the semigroup S belonging to T(S).

2. The main theorem. The purpose of this section is to prove the following

Theorem 1. Let S be a finite connected tolerance semigroup with identity element. Let $\mathcal{A}$ be any group of automorphisms of S. Then the set K of all generalized fixed-points of $\mathcal{A}$ in S is a connected subsemigroup of S which meets the minimal ideal M(S) of S.

The proof will be carried out in three steps.

(A) The structure of M(S). Let S be any finite semigroup, L and R any of its minimal left and right ideals. It is well known that LR equals the least ideal M = M(S) of S and that RL = $L \cap R$ = G is a group. Set X = $L \cap E(S)$ and Y = = $R \cap E(S)$ where E(S) is the set of all idempotents in S. Then it is known and easy to prove that X is a left zero semigroup, Y a right zero semigroup and that we have a direct decomposition

$$M_C = X_C \times G_C \times Y_C$$

Now, assume that S is a finite tolerance semigroup. This assumption makes X, G and Y tolerance semigroups (with tolerances induced by that of S). Moreover, it is easy to show that the above direct decomposition remains true for tolerance spaces M, X, G and Y: for $x, x' \in X$, $g, g' \in G$, $y, y' \in Y$ we have $(xgy)\ t\ (x'g'y')$ if and only if $x\ t\ x'$, $g\ t\ g'$, $y\ t\ y'$.

Remark. A complete description of all finite simple tolerance semigroups is an obvious consequence of our considerations.

The symbols X, G, Y will keep their meanings also in the next section. The projection $M \longrightarrow Y$ coming out of the direct decomposition of M (and which was shown to be a tolerance mapping) will be denoted by π.

(B) <u>The</u> p-<u>contractibility of</u> S <u>and</u> M(S). We start by observing that a tolerance space S is p-contractible if and only if $1_S\ \bar{p}\ c$ for the identity mapping $1_S : S \longrightarrow S$ and for some constant mapping $c : S \longrightarrow S$.

<u>Lemma 1</u>. Let S be a finite connected tolerance semigroup with a right identity element u. Then $Y = R \cap E(S)$ is p-contractible.

Proof: For any $s \in S$ let $f_s : Y \longrightarrow Y$ be defined by $f_s(y) = \pi(ys)$ where $y \in Y$ and $\pi : M \longrightarrow Y$ is the projection. Clearly, $f_s \in T(Y)$. If $a\ t\ b$ in S, then $f_a\ p\ f_b$ in T(Y). Take any fixed $b \in Y$. As $u\ \bar{t}\ b$ by connectedness of S it follows that $f_u\ \bar{p}\ f_b$. But $f_u = 1_Y$ and f_b is constant.

<u>Lemma 2</u>. Let S be a finite connected tolerance semigroup with identity element 1. Then M(S) and S are p-contractible.

Proof: By Lemma 1 we get that Y is p-contractible and we get that X is p-contractible by a dual statement.

Next we shall see that G is connected. For any $a,b \in G$ we have $a\ \bar{t}\ b$ in S and $a\ t\ a_1, a_1\ t\ a_2, \dots, a_n\ t\ b$ for some a_i in S. But we can suppose that all a_i belong to G as every a_i can be replaced by ea_ie with e being the identity element of G.

Now, G is a group and so every tolerance on G is a congruence relation. As G is connected it is a simplex in S and, consequently, G is p-contractible. It follows that $M = X \times G \times \times Y$ is p-contractible.

For any $s \in S$ let $g_s : S \to S$ be defined by $g_s(x) = xs$ for all $x \in S$. Clearly, $g_s \in T(S)$. If $a\ t\ b$ in S, then $g_a\ p\ g_b$ in T(S). Take any fixed $b \in M$. As $1\ \overline{t}\ b$ by connectedness of S it follows that $g_1\ \overline{p}\ g_b$. But $g_1 = 1_S$ and $g_b : S \to M$. As M is p-contractible, $g_b\ \overline{p}\ c$ for some constant $c : S \to M$. Thus $1_S\ \overline{p}\ c$ and S is p-contractible.

(C) <u>The final proof</u>. In this section we make use of the tools and ideas developed in [3]. First we want to recall, for reader's convenience, the main lines of the proof that in a finite p-contractible tolerance space S there is always a non-empty simplex A fixed under all injective $\alpha \in T(S)$ (see [3]).

For any $x \in S$ set $tx = \{y \in S \mid y\ t\ x\}$. Let P be the set of all tx ($x \in S$). P is partially ordered by inclusion relation. Let $D(S) = \{y \in S \mid ty$ is maximal in $P\}$. If $D(S) \neq S$ we continue by taking $D^2(S) = D(D(S))$ and by repeating this procedure until we get a finite descending chain $S \supset D(S) \supset D^2(S) \supset \ldots$ $\ldots \supset D^n(S) = A$ such that $D(A) = A$. Now we have

<u>Lemma 3</u>. D(S) is a retract of S.

This is shown by proving $h \circ j = 1_{D(S)}$ where $j : D(S) \to S$ is the inclusion mapping and $h : S \to D(S)$ is defined as follows: if $x \in D(S)$, set $h(x) = x$; if $x \notin D(S)$, set $h(x) =$ any $y \in D(S)$ with $tx \subset ty$. Let us point out that h is a tolerance mapping: assuming $x\ t\ x'$, $h(x) = y$, $h(x') = y'$, we have $tx \subset ty$, $tx' \subset ty'$, $x' \in tx \subset ty$, $y\ t\ x'$, $y \in tx' \subset ty'$, $y\ t\ y'$.

Lemma 4. D(S) is p-contractible.

This is easy: $1_S \overline{p}\ c$ implies $(h \circ 1_S \circ j)\ \overline{p}\ (h \circ c \circ j)$ and so $1_{D(S)}\ \overline{p}\ c'$.

It follows that $A = D^n(S)$ is p-contractible.

Lemma 5. x) All $ta (a \in A)$ are equal. A is a simplex in S.

Assume that not all $ta\ (a \in A)$ are equal. We have $1_A\ \overline{p}$ const. Consequently, there are tolerance mappings $f, g \in T(A)$ such that (i) $tx = tf(x)$ for all $x \in A$, (ii) there is some $x \in A$ with $tx \neq tg(x)$, (iii) f p g. For the x from (ii) we shall prove $tx \subset ty$ where $y = g(x)$. Really, take any $x' \in tx$, $x'\ t\ x$. As f p g, we conclude $f(x')\ t\ g(x)$, $f(x')\ t\ y$, $y \in tf(x') = tx'$, $x' \in ty$. But tx is maximal as $D(A) = A$, thus $tx = ty$, a contradiction in view of (ii).

Remark. Lemmas 3, 4 and 5 come essentially from [3]. We have made only slight adaptations of the original proofs.

Now we come to the proof of Theorem 1. We assume that S is a finite connected tolerance semigroup with identity element 1 and that $\mathcal{A}$ is any group of automorphisms of S. We denote by K the set of all generalized fixed-points of $\mathcal{A}$ in S. It is clear that the simplex $A = D^n(S)$ is fixed under all $\alpha \in \mathcal{A}$ and thus $A \subset K$. We shall show that K is connected.

Choose any $x_0 \in K$. Obviously, there is some simplex A_0 fixed under $\mathcal{A}$ containing x_0. Set $A_1 = \{y \in D(S) \mid \exists\ x \in A_0 \text{ with } tx \subset ty\}$. It is easy to see that $A_1 \neq \emptyset$ and $A_1 = \alpha(A_1)$ for all $\alpha \in \mathcal{A}$. Moreover, if $y, y' \in A_1$, $x, x' \in A_0$, $tx \subset ty$, $tx' \subset ty'$,

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x) The sets $ta\ (a \in A)$ and all sets of this form in the proof are taken in A.

then $x' \in tx \subset ty$, $y\ t\ x'$, $y \in tx' \subset ty'$, $y\ t\ y'$. It follows that A_1 is a non-empty simplex fixed under $\mathcal{A}$, $A_1 \subset D(S)$. The above lines also show that $x'\ t\ y$ for all $x' \in A_0$ and for all $y \in A_1$. Repeating this construction we obtain non-empty simplices $A_2, A_3, \dots, A_n \subset A$ fixed under $\mathcal{A}$ such that $A_i \subset$ $\subset D^i(S)$ and such that $A_i \times A_{i+1} \subset t$ for all $i < n$. Consequently, there is some $y_0 \in A$ with $x_0\ \bar{t}\ y_0$ in K. K is connected.

If A_0, B_0 are simplices in S fixed under $\mathcal{A}$, then $A_0 B_0$ is a simplex fixed under $\mathcal{A}$. This shows that K is a subsemigroup.

$M = M(S)$ is preserved under all $\alpha \in \mathcal{A}$. If we start the construction of A with M instead of S, we obtain some non-empty simplex A_M in M fixed under $\mathcal{A}$. This means that $A_M \subset K$ and K meets M. The theorem is proved.

3. Further results. Theorem 1 can be essentially supplemented by the following

Statement: Under the assumptions of Theorem 1 there is a connected commutative and idempotent subsemigroup C in K containing the identity element 1 of S and meeting the ideal M(S).

This follows from the next

Theorem 2. Let S be a finite tolerance semigroup. Then the following conditions are equivalent:

(i) for every $e \in E(S) \setminus M(S)$ e is connected with M(S) in S

(ii) for every $e \in E(S) \setminus M(S)$ e is connected with $eSe \setminus H(e)$ in eSe

(iii) for every $e \in E(S) \setminus M(S)$ e is connected with $SeS \setminus D(e)$ in SeS

(iv) for every $e \in E(S) \setminus M(S)$ there is a connected commutative and idempotent subsemigroup C in S containing e and meeting the ideal M(S).

Remarks: We say that e is connected with some $X \subset S$ if and only if $\bar{t}e$ meets X, $\bar{t}e \cap X \neq \emptyset$.

For any $x \in S$ let J(x) denote the ideal generated by x. For reader's convenience we recall that for any $e \in E(S)$, SeS= = J(e), eSe is the set of all $x \in S$ with xe = ex = x, H(e) is the maximal group in S containing e and D(e) is the set of all $x \in S$ with J(x) = J(e). We have $H(e) \subset eSe \subset SeS$ and $D(e) \subset$ $\subset SeS$.

Proof: (iv) implies (i): obvious.

(i) implies (ii): Take $e \in E(S) \setminus M(S)$. We have $e \,\bar{t}\, m$ in S for some $m \in M(S)$. It follows that $e = e^3 \,\bar{t}\, eme$ in eSe and it remains to prove that $eme \notin H(e)$. But $eme \in H(e) \cap M(S)$ implies $H(e) \cap M(S) \neq \emptyset$ and $e \in M(S)$, a contradiction.

(ii) implies (iv): Take $e \in E(S) \setminus M(S)$. We have $e \,\bar{t}\, x$ in eSe for some $x \in eSe \setminus H(e)$. As H(e) is a group and, consequently, t induces a congruence relation on H(e), we can suppose that e t y, y t x for some $y \in H(e)$. But then $e = (y^{-1}y)\ t\ (y^{-1}x)$ and $y^{-1}x \in eSe \setminus H(e)$. Hence, making a better choice of x, we can suppose that e t x, $x \in eSe \setminus H(e)$. Consequently, $x\ t\ x^2$ and, in general, $x^n\ t\ x^{n+1}$ for all n = 1,2,3,... . There is some k such that $j = x^k \in E(S)$. As $x \notin H(e)$ we have $j \neq e$. As $ej = je = j$ we conclude that $e > j$. From e t x we get $e^k\ t\ x^k$, e t j.

Repeating this procedure we obtain a descending chain in E(S) $e > j_1 > j_2 > j_3 > \ldots$ with $j_n\ t\ j_{n+1}$ (n = 1,2,3,...) which must terminate with some $j_s \in M(S)$. We set C =

$= \{e, j_1, j_2, \dots, j_s\}$.

<u>(iii) implies (i)</u>: Take any $e \in E(S) \setminus M(S)$. Then $e \in D(e)$ and $e \,\bar{t}\, a$ for some $a \in J(e) \setminus D(e)$. From $e \,\bar{t}\, a$ follows easily $e^n \,\bar{t}\, a^n$ for all $n = 1,2,3,\dots$. There is some k such that $a^k = e_1 \in E(S)$. We have $e \,\bar{t}\, e_1$ and $J(e_1) \subset J(a) \subsetneqq J(e)$. If $e_1 \notin M(S)$, we continue this procedure and we get finally a sequence $e, e_1, e_2, \dots$ in $E(S)$ with $e_n \,\bar{t}\, e_{n+1}$ $(n = 1,2,3,\dots)$. As $J(e) \supsetneqq J(e_1) \supsetneqq J(e_2) \supsetneqq \dots$ the sequence must be finite. We have some $e_s \in M(S)$.

<u>(i) implies (iii)</u>: Take $e \in E(S) \setminus M(S)$. Then $e \,\bar{t}\, m$ in S for some $m \in M(S)$. It follows that $e^2 \,\bar{t}\, em$ and $e \,\bar{t}\, em$ in SeS. We have to prove yet $em \notin D(e)$. But $em \in D(e)$ implies $J(em) = J(e)$, $e \in J(em) \subset M(S)$, a contradiction.

<u>Remark</u>. In the condition (iv) of Theorem 2 the connected semilattice C can be replaced by a connected chain as shown in the proof of (ii) $\Longrightarrow$ (iv). The same can be remarked about the statement before Theorem 2. As to the proof of this statement we observe that $1 \in K$ and that 1 is connected in K with M(K). It follows easily that K satisfies the condition (i) of Theorem 2 and, consequently, condition (iv).

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Carolinae 4(1963), 121-136.

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(Oblatum 21.1. 1980)