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THE AMALGAMATION PROPERTY OF VARIETIES DETERMINED BY PRIMITIVE LATTICES

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Abstract: No variety determined by a primitive lattice has the Amalgamation Property.

Key words: Lattice, primitive lattice, variety, the Amalgamation Property.

Classification: 06A20

A class K of lattices is said to have the Amalgamation Property if, whenever $A, B, C \in K$ are lattices such that C is a sublattice of both A and B, then there is a lattice $Z \in K$ and embeddings f of A into Z and g of B into Z such that $f(c) = g(c)$ for all $c \in C$.

Let L be a lattice. Denote by N(L) the class of all lattices that contain no sublattice isomorphic to L. A lattice L is said to be primitive if N(L) is a variety. The complete description of all primitive lattices is given in [1]; the reader is supposed to be acquainted with [1].

The aim of this note is to show that no variety V = = N(L) (where L is a primitive lattice) has the Amalgamation Property.

Let us remark that both extreme varieties of lattices and the variety of distributive lattices have the Amalgamation Property; it is an open problem (cf. [2]) to determine the number of varieties of lattices with the Amalgamation Property.

A lattice L is said to be A-decomposable if there exist proper sublattices L_1, L_2 of L such that whenever f_i ($i = 1,2$) are embeddings of L_i into a lattice Z and $f_1(x) = f_2(x)$ for all $x \in L_1 \cap L_2$ then L can be embedded into Z.

Let L_1, L_2 be proper sublattices of a lattice L. We shall say that the condition $P_\vee(L_1,L_2)$ is satisfied if $L_1 \cup L_2 = L$ and for all $x \in L_1 \setminus L_2$, $y \in L_2 \setminus L_1$ one of the following conditions is satisfied:

1) there exists a $c \in L_1 \cap L_2$ such that either $c \leq x$ and $c \vee y \in L_1 \cap L_2$ or $c \leq y$ and $c \vee x \in L_1 \cap L_2$.

2) there exist $c,d \in L_1 \cap L_2$ such that either $c \leq x \leq d \leq c \vee y$ or $c \leq y \leq d \leq x \vee c$.

3) there exists a $c \in L_1 \cap L_2$ such that either $x \leq c \leq y$ or $y \leq c \leq x$. The condition $P_\wedge(L_1,L_2)$ is defined dually.

<u>Lemma 1</u>. Let L_1, L_2 be proper sublattices of a lattice L and let $P_\vee(L_1,L_2)$ and $P_\wedge(L_1,L_2)$ be satisfied. Then L is A-decomposable.

Proof. Let f_i ($i = 1,2$) be embeddings of L_i into a lattice Z such that $f_1(x) = f_2(x)$ for all $x \in L_1 \cap L_2$. We shall show that the mapping $h = f_1 \cup f_2$ is an embedding of L into Z. First we shall prove that h is injective. Let $x \neq y$ and $h(x) = h(y)$. It is enough to assume that $x \in L_1 \setminus L_2$ and $y \in L_2 \setminus L_1$.

Case 1: $c \in L_1 \cap L_2$, $c \leq x$ and $c \vee y \in L_1 \cap L_2$. Then $f_2(y) = h(y)=h(x)=f_1(x)=f_1(c \vee x)=f_1(c) \vee f_1(x)=f_2(c) \vee f_1(x)$.

We have $f_2(c) \leq f_2(y)$ and so $c \leq y = y \vee c \in L_1 \cap L_2$; a contradiction.

Case 2: $c, d \in L_1 \cap L_2$ and $c \leq x \leq d \leq c \vee y$. Then $f_1(x) = f_1(c) \vee f_1(x) \leq f_1(d) = f_2(d) \leq f_2(c \vee y) = f_2(c) \vee f_2(y) = f_1(c) \vee \vee f_1(x) = f_1(x)$.

We have $f_1(x) = f_1(d)$ and so we get $x = d \in L_1 \cap L_2$; a contradiction.

Case 3: $c \in L_1 \cap L_2$ and $x \leq c \leq y$. Then $h(x) = f_1(x) \leq f_1(c) = f_2(c) \leq f_2(y) = h(y) = h(x)$.

We have $f_1(x) = f_1(c)$ and so $x = c \in L_1 \cap L_2$; a contradiction.

Now we shall prove that h is a homomorphism. It is enough to verify $h(x \vee y) = h(x) \vee h(y)$ for all $x \in L_1 \setminus L_2$, $y \in \in L_2 \setminus L_1$.

Case 1: $c \in L_1 \cap L_2$, $c \leq x$ and $y \vee c \in L_1 \cap L_2$. Then $h(x \vee y) = h(c \vee x \vee y) = f_1(c \vee x \vee y) = f_1(x) \vee f_1(c \vee y) = f_1(x) \vee \vee f_2(c \vee y) = f_1(x) \vee f_2(c) \vee f_2(y) = f_1(x) \vee f_1(c) \vee f_2(y) = f_1(x) \vee f_2(y) = h(x) \vee h(y)$.

Case 2: $c, d \in L_1 \cap L_2$ and $c \leq x \leq d \leq c \vee y$. Then $h(x \vee y) = h(c \vee x \vee y) = h(c \vee y) = f_2(c \vee y) = f_2(c) \vee f_2(y) = f_1(c) \vee f_2(y) \leq \leq f_1(x) \vee f_2(y) = h(x) \vee h(y)$. $h(y) = f_2(y) \leq f_2(c \vee y) = h(x \vee y)$.
$h(x) = f_1(x) \leq f_1(d) = f_2(d) \leq f_2(c \vee y) = h(x \vee y)$.
So we get $h(x) \vee h(y) = h(x \vee y)$.

Case 3: $c \in L_1 \cap L_2$ and $x \leq c \leq y$. Then $h(x) \vee h(y) = f_1(x) \vee \vee f_2(y) = f_1(x) \vee f_2(c \vee y) = f_1(x) \vee f_2(c) \vee f_2(y) = f_1(x) \vee f_1(c) \vee \vee f_2(y) = f_1(c) \vee f_2(y) = f_2(c) \vee f_2(y) = f_2(y) = h(x) = h(x \vee y)$.

Let A_2, A_3, A_4, B_n $(n \geq 1)$, C_n $(n \geq 1)$, D_n $(n \geq 0)$, E_n $(n \geq 0)$, F_n $(n \geq 2)$, G_n $(n \geq 2)$ be the same lattices as the lattices defined and pictured in [1] and let R, P, Q denote the same constructions as those defined in [1].

Lemma 2. The lattices A_2, A_3, A_4, B_n $(n \geq 1)$, C_n $(n \geq 1)$ are A-decomposable.

Proof. Let $L \in \{A_2, A_3, A_4, B_n, C_n\}$. The lattice L has exactly two both meet and join irreducible elements a, b. Put $L_1 = L \setminus \{a\}$, $L_2 = L \setminus \{b\}$. It is easy to verify the conditions $P_{\vee}(L_1, L_2)$ and $P_{\wedge}(L_1, L_2)$.

Lemma 3. Let L be a lattice of cardinality at least 3. Then the lattice R(L) is A-decomposable. If, moreover, there exist elements a, $t \in L$ such that $a \neq 0_L, 1_L$ (the least and the greatest element of L) and such that $L = (a] \cup [t)$ (the disjoint union), then the lattices P(L,a) and Q(L,a) are A-decomposable.

Proof. Put $L_1 = R(L) \setminus \{c_L\}$, $L_2 = \{0_L, 1_L, c_L, o_L, i_L\}$. Put $L_1 = P(L,a) \setminus \{c_L\}$, $L_2 = \{1_L, i_L, c_L, a, t, t \vee a, t \wedge a\}$. Put $L_1 =$ $= Q(L,a) \setminus \{d_L\}$, $L_2 = \{1_L, i_L, c_L, d_L, o_L, 0_L, a, t, a \vee t, a \wedge t\}$. The verification of $P_{\vee}(L_1, L_2)$ and $P_{\wedge}(L_1, L_2)$ is easy.

Lemma 4. The lattices D_n' $(n \geq 0)$, E_n' $(n \geq 0)$, F_n' $(n \geq 2)$, G_n' $(n \geq 2)$ pictured in Fig. 1 are A-decomposable.

Proof. Let $L \in \{D_n', E_n', F_n', G_n'\}$. It is a mechanical work to verify that the conditions $P_{\vee}(L_1, L_2)$, $P_{\wedge}(L_1, L_2)$ are satisfied for the sublattices $L_1 = L \setminus \{a, b\}$ and $L_2 = (k]$ (the ideal generated by k) where a, b, k are the elements pictured in Fig. 1.

Let T be the class of all lattices L such that the class N(L) does not have the Amalgamation Property. It is evident that any finite A-decomposable lattice belongs to T and so we get from Lemma 1 that the lattices A_2, A_3, A_4, B_n, C_n

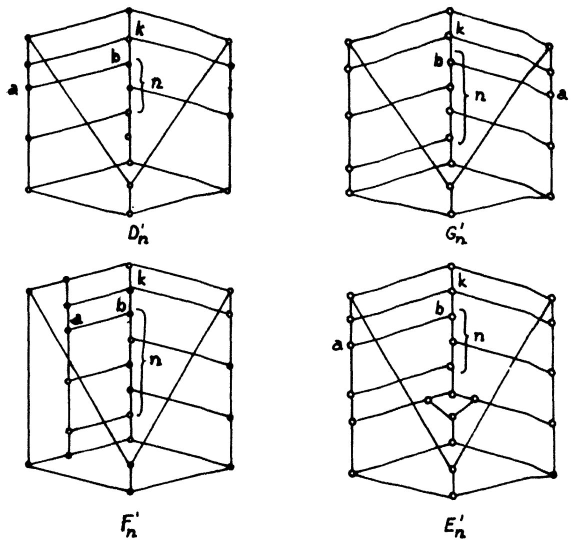

Figure 1.

belong to T. Since the lattices D_n', E_n', F_n', G_n' are A-decomposable into two sublattices not containing a sublattice isomorphic to D_n, E_n, F_n or G_n, respectively and the lattices D_n, E_n, F_n, G_n can be embedded into D_n', E_n', F_n' and G_n', respectively, we get that the lattices D_n, E_n, F_n, G_n are in T. If L is a finite lattice having at least three elements, then by Lemma 3 the lattice R(L) belongs to T; if, moreover, $L = (a] \cup [t)$ (the disjoint union) for some $a, t \in L$, then the lattices P(L,a) and Q(L,a) belong to T. Evidently T is closed under the dual lattices. Combining the facts mentioned above with the main result of [1] we get that all primitive lattices except for the two-element lattice and the five-element nonmodular lattice are in T. Since the class of all modular lattices does not have the Amalgamation Property [2], we get

Theorem. Let V be a nontrivial variety of lattices and let there exist a lattice L such that V is the class of all lattices that do not contain a sublattice isomorphic to L. Then V does not have the Amalgamation Property.

R e f e r e n c e s

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