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Eric K. Douwen

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NONSUPERCOMPACTNESS AND THE REDUCED MEASURE ALGEBRA

Eric K. van DOUWEN[1])

Abstract: Easy known results easily imply that the Stone space of the reduced measure algebra is not supercompact, and in fact is not n-supercompact whenever $3 \leq n < \omega$.

Key words: n-supercompact, n-linked, Stone space, measure algebra, σ-n-linked, separable

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What we do: Supercompact spaces, defined below, (and more generally n-supercompact spaces, $3 \leq n < \omega$) are compact. Compact linearly orderable spaces are easy examples of supercompact spaces. Compact metrizable spaces are supercompact, [SS] (see [vD_2] and [M_1] for easier proofs), and so are compact groups, [M_2].

An easy example of a nonsupercompact compact space was given by Verbeck, [V, II.2.2]; this example is T_1 but not Hausdorff. The first Hausdorff examples were given by Bell, [B_1]; other examples, or other proofs, can be found in the references.

The purpose of this note is to present an essentially trivial very natural Hausdorff example.

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EXAMPLE: The Stone space of the reduced measure algebra is not supercompact, and in fact is not n-supercompact for any n with $3 \leq n < \omega$.

For the proof we only need easy known results about the reduced measure algebra, and closed subbases (defined below), or easy modifications thereof (we include proofs for completeness sake), and unlike other examples, need only an easy fact about (n-) supercompactness.

Closed subbases: A family S of subsets of a space X is called a *closed subbase* for X if $\{X - S: S \in S\}$ is a subbase for the open sets of X, i.e. if for every nonempty $F \subset X$, F is closed iff $F = \cap_{F \in \underline{C}} \cup F$ for some collection $\underline{C}$ of finite subfamilies of S. We need the following elementary fact, where $^\circ$ denotes the interior operator.

FACT 1: If X is compact Hausdorff, and S is a closed subbase for X, then for each nonempty open $U \subseteq X$ *there is a (nonempty)* $G \subseteq S$ *with* $\cap G \subseteq U$ *and* $(\cap G)^\circ \neq \emptyset$.

□ Pick a nonempty open $V \subseteq X$ with $\overline{V} \subseteq U$. Let $\underline{C}$ be a (nonempty) collection of finite subfamilies of S with $\cap_{F \in \underline{C}} \cup F = \overline{V}$. Since X is compact there is a (nonempty) finite $\underline{F} \subseteq \underline{C}$ with $\cap_{F \in \underline{F}} \cup F \subseteq U$. Clearly $\cap_{F \in \underline{F}} \cup F = \cup_{G \in \underline{G}} \cap G$ for some finite collection $\underline{G}$ of (nonempty) (finite) subfamilies of S. There is $G \in \underline{G}$ with $(\cap G)^\circ \neq \emptyset$ since $(\cup_{G \in \underline{G}} \cap G)^\circ \supseteq V \neq \emptyset$. □

n-Supercompactness: For a cardinal κ call a family F of sets κ-*linked* if $\cap G \neq \emptyset$ for every $G \subseteq F$ with $0 < |G| < \kappa$, so 3-linked≡linked, and ω-linked≡centered. Also, call a space κ-*supercompact* if it has a closed subbase S such that every nonempty κ-linked subfamily of S has nonempty intersection. Clearly ω-supercompact≡compact, by Alexanders subbase

Theorem, [K, p. 139], and κ-supercompact implies λ-supercompact if $\kappa \leq \lambda$ (but not conversely if $3 \leq \kappa < \lambda \leq \omega$, [BvM]). So 3-supercompact is supercompact, as introduced by de Groot, [dG], and a space is n-supercompact if it has compactness number $<n$, as defined in [BvM].

The reduced measure algebra. Let I be the closed unit interval, let M be the Boolean algebra of measurable subsets of I, and let N be the ideal of null-sets. The quotient algebra M/N is called the *reduced measure algebra*. Let $\mathbb{M}$ denote its Stone space.

Let λ denote Lebesgue measure on I, and for $A \in M$ let $[A]$ denote the N-equivalence class of A.

Call a family σ-κ-*linked* if it is the union of countably many κ-linked families.

FACT 2: *The family of nonempty clopen (≡ closed and open) subsets of $\mathbb{M}$ is σ-n-linked for each n with $3 \leq n < \omega$.*

[] We prove the corresponding statement for $M - N$. Fix n with $3 \leq n < \omega$.

Let $\mathcal{B}$ be a countable base for I with $\emptyset \notin \mathcal{B}$ which is closed under finite union. For $B \in \mathcal{B}$ define

$$L_B = \{A \in M\colon \lambda(A \cap B) > (1 - n^{-1})\cdot\lambda(B)\}.$$

Clearly, if $\mathcal{A} \subset L_B$ and $0 < |\mathcal{A}| < n$ then $\lambda(\cap\mathcal{A}) \geq \lambda(\cap\mathcal{A} \cap B) > n^{-1}\lambda(B) > 0$, so $\cap\mathcal{A} \notin N$. Next, given $A \in M - N$ find compact $K \subseteq A$ with $\lambda(K) > 0$, and then find $B \in \mathcal{B}$ with $B \supseteq K$ and $\lambda(K) < (1-n^{-1})^{-1}\cdot\lambda(B)$. Then $K \in L_B$, hence $A \in L_B$.

[This (fact and proof) is well known for $n = 3$ of course.] []

FACT 3: *$\mathbb{M}$ is not separable.*

□ Let $\langle p_n \rangle_n$ be any sequence in M. For $n < \omega$, since p_n is an ultrafilter in the Boolean algebra M/N we can pick $P_n \in M - N$ with $[P_n] \in p_n$ and $\lambda(P_n) < 2^{-2-n}$. Then $\{p \in M: [I - \cup_n P_n] \in p\}$ is a nonempty open set in M that contains no p_n. [This is known of course.] □

<u>Separability and supercompactness</u>: The following result implies that M is not n-supercompact, because of Facts 2 and 3, since the family of clopen sets of M is a base.

FACT 4: *Let $3 \leq \kappa \leq \omega$. Then following conditions on a κ-supercompact Hausdorff space X are equivalent:*

(1) *X is separable;*

(2) *the topology of X is σ-centered ($\equiv$ σ-ω-linked); and*

(3) *X has a σ-κ-linked base (or π-base).*

□ We prove (3) $\Rightarrow$ (1). Let S be a closed subbase for X that witnesses that X is κ-supercompact. Let B_n be a κ-linked family of open sets of X for $n < \omega$ such that $\cup_n B_n$ is a base for X. We can assume $\emptyset \neq X \in S$, and $\emptyset \notin \cup_n B_n$. Then for each $n < \omega$ the family

$$S_n = \{S \in S: \exists B \in B_n (B \subseteq S)\}$$

is nonempty and κ-linked, hence we can pick $p_n \in \cap S_n$. It now follows from Fact 1 that $\{p_n: n < \omega\}$ is dense in X. [For $\kappa = \omega$ this is known, [vD$_1$].] □

QUESTION 1. Does there exist for each n with $3 \leq n < \omega$ a nonseparable $(n+1)$-supercompact Hausdorff space whose topology is σ-n-linked? Or at least a compact Hausdorff space whose topology is σ-n-linked but not σ-$(n+1)$-linked?

QUESTION 2. Is Fact 4 true if X is a Hausdorff continuous image of a κ-supercompact Hausdorff space?

The motivation for Question 2 is that although supercompactness is not preserved by continuous maps, [MvM], most results about supercompact Hausdorff spaces are true for Hausdorff continuous images of (closed neighborhood retracts) of supercompact Hausdorff spaces, hence a counter example to Question 2 for $\kappa = 3$ would be a nice example that supercompactness is not preserved under continuous maps.

In this context we point out that M is in fact not a continuous image of a (closed neighborhood retract of a) Hausdorff space which is n-supercompact for some n with $3 \leq n < \omega$ (see also [BvM]). For the proof one notes that Fact 4 is true if X is a closed neighborhood retract of a κ-supercompact Hausdorff space [use the proof of Fact 1 rather than Fact 1 itself], and that M is extremally disconnected, so that a compact Hausdorff space has a retract homeomorphic to M iff it can be mapped onto M, [G, Thm. 2.5].

References

[B_1] M.G. Bell, *Not all compact spaces are supercompact*, Gen. Top. Appl. 8 (1978), 151-155.

[B_2] ________, *A cellular constraint in supercompact Hausdorff spaces*, Can. J. Math. 30 (1978), 1144-1151.

[B_3] ________, *A first countable supercompact Hausdorff space with a closed G_δ non-supercompact subspace*, Coll. Math. (to appear).

[BvM] M.G. Bell and J. van Mill, *The compactness number of a compact topological space*, Fund. Math. (to appear).

[vD_1] E.K. van Douwen, *Density of compactifications*, in 'Set theoretic topology', G.M. Reed (ed.), Academic Press, New York (1977), 97-110.

[vD_2] ________, *Special bases for compact metrizable spaces*, Fund. Math. (to appear).

[vDvM] E.K. van Douwen and J. van Mill, *Supercompact spaces*, Top. Appl. (to appear).

[G] A.M. Gleason, *Projective topological spaces*, Ill. J. Math. 2 2 (1958), 482-489.

[dG] J. de Groot, *Supercompactness and superextensions*, Contrib. to Extension Theory of Top. Struct. Symp. Berlin 1967, Deutscher Verlag Wiss., Berlin (1969), 89-90.

[K] J.L. Kelley, *General Topology*, Van Nostrand Reinhold Cy., New York, 1955.

[vMM_1] J. van Mill and C.F. Mills, *On the character of supercompact topological spaces*, Top. Proc. 3 (1978), 227-236.

[vMM_2] ________, *Closed G_δ subsets of supercompact Hausdorff spaces*, Indag. Math. 41 (1979), 155-162.

[M_1] C.F. Mills, *A simpler proof that compact metric spaces are supercompact*, Proc. AMS 73 (1979), 388-390.

[M_2] ________, *Compact topological groups are supercompact*, Fund. Math. (to appear).

[MvM] C.F. Mills and J. van Mill, *A nonsupercompact continuous image of a supercompact space*, Houston J. Math. 5 (1979), 241-247.

[SS] M. Strok and A. Szymański, *Compact metric spaces have binary bases*, Fund. Math. 89 (1975), 81-91.

[V] A. Verbeek, *Superextensions of topological spaces*, Ph.D. dissertation, Univ. of Amsterdam, 1972 = Mathematical Centre Tract 41, Amsterdam, 1972.

Institute for Medicine and Mathematics,

Ohio University,

Athens, OH 45701

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