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Selections using orderings (non-separable case)

Commentationes Mathematicae Universitatis Carolinae, Vol. 21 (1980), No. 4, 653--661

Persistent URL: http://dml.cz/dmlcz/106032

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Abstract: Two selection theorems with their proofs using the lexicographic ordering of sequences of positive integers are extended for correspondences of complete (non-separable) metric spaces.

Key words: Point-analytic space, point-Luzin space, Suslin set, Baire set, $\mathcal{C}$-dd-preserving correspondence, $\mathcal{C}$-db-preserving correspondence.

Classification: 54C65, 54H05

The main result is Theorem below which generalizes the selection Theorem of von Neumann [N] and partially the "uniformization type" Theorem of Mazurkiewicz [M] to the non-separable case. The proofs follow the pattern of the proofs of von Neumann (Lemma 1(a) corresponds to [N, Lemma 16]) and of K. Kuratowski (Lemma 1(b) corresponds to [K, Th. 3, p. 491]) respectively. Lemma 1(b) is proved also in [Hol].

The proofs are using the lexicographic order on $\aleph^\omega$. In what follows, $\aleph$ is an infinite cardinal conceived as the set of all ordinals of cardinal $<\aleph$, and endowed with its well order and the discrete uniformity. The product space $\aleph^\omega$ is a metrizable complete uniform space endowed with the lexicographic order $\downarrow$ defined as follows:
for the smallest $k$ such that $\alpha_k \neq \beta_k$.

In what follows, we shall need the following two elementary facts about the order: each non-void closed set in $\mathcal{E}^\omega$ has the smallest element, and the set

$$\{ \langle d_1, d_2 \rangle \mid d_1 \preceq d_2 \}$$

is closed in the product space $\mathcal{E}^\omega \times \mathcal{E}^\omega$.

On the other hand, we need to know the concepts of analytic, point-analytic and point-Luzin spaces, and several basic properties from [H1, 2]. Also, the term Baire set is used for a more general notion (corresponding to extended Borel of Hansell in metric spaces, and to hyper-Baire used by the first author in his earlier work). For convenience of the reader we quickly recall what is needed.

By a space we always mean a uniform space, and the topologically fine uniformity (called fine by J. Isbell) consists of all continuous pseudometrics. "Discrete" is understood in the uniform sense. A family $\{ X_a \mid a \in A \}$ is called to be $\sigma$-discretely decomposable (abb. $\sigma$-dd) if there exists a family $\{ X_{an} \mid a \in A, n \in \omega \}$ such that each family $\{ X_{an} \mid a \in A \}$ is discrete, and $X_a = \bigcup \{ X_{an} \mid n \in \omega \}$ for each $a$. A family $\{ X_a \}$ is said to be $\sigma$-db ($\sigma$-discretely base-like refinable) if there exists a $\sigma$-discrete collection $\mathcal{B}$ such that each $X_a$ is the union of a subfamily of $\mathcal{B}$. Clearly $\sigma$-dd implies $\sigma$-db. For Lemma 2 we need to know that in a metric space $\{ X_a \}$ is $\sigma$-dd or $\sigma$-db iff it is such in the fine uniformity, and locally $\sigma$-dd implies $\sigma$-dd (see [H, Lemma 2 and Corollary 1]).

A correspondence $F = \text{gr } F : X \rightarrow Y$ (gr $F \subset X \times Y$) is said to
be $\sigma'$-dd-preserving or $\sigma$-db-preserving provided that if 
$\{X_a\}$ is $\sigma'$-dd or $\sigma$-db in $X$, then so is $\{F[X_a]\}$ in $Y$. To 
check the properties, it is enough to check the images of dis-
crete families $\{X_a\}$.

A space $X$ is called point-analytic if there exists a 
continuous $\sigma$-dd-preserving mapping of a complete metric 
space $P$ onto $X$; if $f$ may be chosen 1-1, then $X$ is called point-
Luzin. One obtains the definition of analytic if $f$ is allowed 
to be an upper-semicontinuous compact-valued correspondence.

We need to know that if $X \subseteq Y$ and $X$ is point-analytic,
then $X$ is Suslin in $Y$ (derived by the Suslin operation from 
the closed sets of $X$) - [FH$_1$, Corollary 4.3(a)], and if $X$ is 
Suslin in $Y$ and $Y$ is point-analytic, then so is $X$ [FH$_2$, Corol-
larly 3.4].

We also need to know that if $f : X \rightarrow Y$ is a surjective 
continuous $\sigma$-db-preserving mapping, then $Y$ is point-analytic 
wenever $X$ is, and if $f$ is moreover injective, then $Y$ is 
point-Luzin whenever $X$ is [FH$_2$, Th. 3.6(a)].

We denote by $\text{Ba}(X)$ (see [FH$_2$, § 1.1]) the smallest $\sigma$-
algebra containing the zero sets of uniformly continuous func-
tions, that is closed under the operation of taking arbitrary 
discrete unions. The elements of $\text{Ba}(X)$ are called Baire sets. 
If we replace "zero sets of uniformly continuous functions" 
by the collection $\mathcal{F}(X)$ of all Suslin sets in $X$, we obtain the 
definition of $\mathcal{F}(X)$.

Finally, a correspondence $F : X \rightarrow Y$ is said to be $(\mathcal{M} \leftarrow \mathcal{N}) 
measurable if $F^{-1}[N] \in \mathcal{M}$ for each $N$ in $\mathcal{N}$.

It should be remarked that Lemma 1(a) can be proved by 
the method of [KRN] and [KP].
1. The purpose of this section is to prove the following result.

**Lemma 1.** Let $P$ be a closed subspace of $\mathbb{R}^\omega$, and let $h$ be a continuous mapping from $P$ onto a uniform space $X$. Consider the selection $s : X \to P$ for $h^{-1}$ such that $s(x)$ is the smallest element of $h^{-1}[x]$ for each $x \in X$. Then:

(a) If $h$ is $\sigma$-db-preserving, then $s$ is $(\overline{\mathcal{F}}(X) \leftarrow \mathcal{B}(P))$-measurable (and of course, $s^{-1}$ is $\sigma$-dd-preserving as the inverse to any selection of $h^{-1}$ is).

(b) If $h$ is $\sigma$-dd-preserving the set $s[X]$ is co-Suslin in $P$ (i.e. $P \setminus s[X]$ is Suslin).

**Proof of (a):** Let $t$ be any selection for $h^{-1}$. If $\{D_a\}$ is a disjoint family in $t[X]$, then $t^{-1}[D_a]$ is disjoint and $h[D_a] = t^{-1}[D_a]$ for each $a$. Hence, if $\{D_a\}$ is discrete, then $\{h[D_a]\}$ is $\sigma$-db, and being disjoint, it is $\sigma$-dd. Thus $t^{-1}$ is $\sigma$-dd-preserving, particularly, $s^{-1}$ is $\sigma$-dd-preserving.

It follows now to show the measurability it is enough to find a $\sigma$-discrete open base $\mathcal{B}$ for the topology of $P$ such that $s$ is $(\overline{\mathcal{F}}(X) \leftarrow \mathcal{B})$-measurable. We take the usual basis consisting of sets of the form

$$B(a) = \{ b \in P | b|n+1 = a \}$$

where $a=(a_0,\ldots,a_n) \in \mathbb{R}^{n+1}$, $n < \omega$, and prove that $s^{-1}[B(a)]$ is the difference of two Suslin sets of $X$. To this end, for each $d \in \mathbb{R}^\omega$ let

$$I(d) = \{ c | c \in \mathbb{R}^\omega, c < d \}.$$

Clearly $I(d)$ is an open set, and it is easy to see that for each finite sequence $a$ ranging in $\mathbb{R}$ there exist $c$ and $d$ in $\mathbb{R}^\omega$ such that
B(a) = I(d) ∩ I(d).

(Put c = {a_0, ..., a_{n-1}, a_n, 0, 0, ...} and d = {a_0, ..., a_{n-1}, a_n + 1, 0, 0, ...}.)

Now the proof is concluded by showing that \( s^{-1}[I(d)] \) is analytic, hence Suslin, for each d. Observe \( s^{-1}[I(d)] = \{ x | s(x) \in d \} = \{ x | \exists c \in h^{-1}[x] \text{ with } c \in d \} = h[I(d)]. \)

Now \( h[I(d)] \) is analytic, because \( I(d) \) is analytic (it is complete metrizable), and \( h \) is a continuous \( \sigma \)-db-preserving mapping \([FH_2, \text{ Th. 3.6(a)}]\).

**Remark.** Without changing the proof, the assumption "\( h \) is \( \sigma \)-db-preserving" in Lemma 1(a) may be weakened to "\( h \) is \( \sigma \)-dr-preserving" whenever we know that the image of an analytic space under a continuous \( \sigma \)-dr-preserving mapping is analytic, and this is actually true. One can do that by a slight modification of the proof for \( \sigma \)-db in \([FH_1, \text{ Th. 4.}].\)

For the proof of Lemma 1(b) we need the following

**Lemma 2.** Let \( h \) be a \( \sigma \)-dd-preserving continuous mapping from a metric space \( P \) onto a uniform space \( X \). Let

\[ M = \{ (d_1, d_2) \in P \times P | h[d_1] = h[d_2] \}. \]

Then the projections

\[ \pi_1 = \{ (x, y) \to x \}: P \times P \to P \]
and
\[ \pi_2 = \{ (x, y) \to y \}: P \times P \to P \]
restricted to \( M \) are \( \sigma \)-dd-preserving.

**Proof of Lemma 2.** Because of symmetry it suffices to prove the assertion for \( \pi_2. \)

Let \( \{ D_a \}_{a \in A} \) be a discrete family in \( M \). There exist \( \sigma \)-discrete open covers \( U \) and \( V \) of \( P \) such that if \( U \in \mathcal{U} \) and \( V \in \mathcal{V} \),
then \((U \times V) \cap D_a \neq \emptyset\) for at most one \(a \in A\). Since \(V\) is \(\sigma\)-discrete and \(P\) is \(\sigma\)-dd-simple, by \([FH_1, \text{Prop. 1.23}]\) it is enough to show that \(V \cap \pi_2[D_a]\) is \(\sigma\)-dd. Clearly \(V \cap \pi_2[D_a] = \pi_2[(P \times V) \cap D_a]\), and the family \(\{\pi_1[(P \times V) \cap D_a]\}\) is discrete because each \(U \in \mathcal{U}\) meets at most one of its members. Thence we may and shall assume that \(\{D_a\}\) is a discrete family in \(M\) such that \(\{\pi_1[D_a]\}\) is discrete in \(P\). Since \(h\) is \(\sigma\)-dd-preserving, the family \(\{h[\pi_1[D_a]]\}\) is \(\sigma\)-dd in \(X\). Since the mappings \(h \circ \pi_1\) and \(h \circ \pi_2\) coincide on \(M\), we have that \(\{h[\pi_2[D_a]]\}\) is \(\sigma\)-dd in \(X\), and since \(h\) is continuous, necessarily

\[(\star) \quad h^{-1}[h[\pi_2[D_a]]]\]

is \(\sigma\)-dd in the fine uniformity of \(P\), and since \(P\) is metric, the family is \(\sigma\)-dd in \(P[Ha, \text{Lemma 2}]\). Finally, \(\{\pi_2[D_a]\}\) is \(\sigma\)-dd because it is dominated by the \(\sigma\)-dd family \((\star)\).

**Proof of Lemma 1(b).** It is easy to check

\[s[x] = P \setminus \pi_1[\{(d_1, d_2) \in P \times P| h[d_1] = h[d_2], d_1 \neq d_2\}]
\]

where \(\pi_1\) is the projection on the first factor. The set in the brackets is equal to

\[\{(d_1, d_2) \in P \times P| h[d_1] = h[d_2]\} \cap \{(d_1, d_2) \in P \times P| d_1 \neq d_2\}.
\]

Since the first set is closed and the second one is open, the intersection is analytic \([FH_2, \text{Th. 3.3}]\), hence the image under \(\pi_1\) is analytic because \(\pi_1\) restricted to \(\{(d_1, d_2)| h[d_1] = h[d_2]\}\) is \(\sigma\)-dd-preserving by Lemma 2. Thus \(s[x]\) is the complement of a Suslin set in \(P\).

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2. Corollaries. The main result reads as follows:

**Theorem.** Let \(F:X \rightarrow Y\) be a correspondence of uniform
spaces $X$ and $Y$. Then:

(a) If the graph $\text{gr} F$ of $F$ is point-analytic and if the projection $\pi_1 : \text{gr} F \to X$ is $\sigma$-db-preserving, then $F$ admits a $(\mathcal{G}(X) \leftarrow \text{Ba}(Y))$-measurable selection $f$.

(b) If $\text{gr} F$ is point-Luzin and if $\pi_1 : \text{gr} F \to X$ is $\sigma$-dd-preserving, then there exists a $(\mathcal{G}(X) \leftarrow \text{Ba}(Y))$-measurable selection $f$ for $F$ such that $\text{gr} F \setminus \text{gr} f$ is analytic.

Proof. (a) Since $\text{gr} F$ is point-analytic, by definition there exists a $\sigma$-dd-preserving continuous mapping $g$ of a closed subspace $P$ of some $\omega_\omega$ onto $\text{gr} F$.

Put $h = \pi_1 \circ g$ and apply Lemma 1(a) (we may suppose that $DF = X$, i.e. $\pi_1[\text{gr} F] = X$) to obtain a $(\mathcal{G}(X) \leftarrow \text{Ba}(Y))$-measurable selection $s$ for $h$. Put $f = \pi_2 \circ g \circ s$. All three maps are $(\mathcal{G} \leftarrow \text{Ba})$-measurable, and so is then $f$.

(b) In this case we may assume that $g$ is a bijection. Lemma 1(b) applies to $h$, and since $g$ is bijective

$g[P \setminus s[X]] = \text{gr} F \setminus \text{gr} f$,

and hence the set is analytic as the image of an analytic space by a continuous $\sigma$-dd-preserving mapping.

We conclude with several consequences of Theorem; in each of the cases the hypothesis would imply that of Theorem. Of course, we need to apply further results.

Corollary 1. There exists a $(\mathcal{G}(X) \leftarrow \text{Ba}(Y))$-measurable selection $f$ for a closed-valued-correspondence $F : X \to Y$ provided that the following three conditions are satisfied:

($\alpha$) $F$ is Suslin measurable (i.e. $(\mathcal{G}(X) \leftarrow \mathcal{G}(Y))$-measurable)

($\beta$) $F^{-1}$ is $\sigma$-dd-preserving
(γ) X is point-analytic and Y is a subspace of a point-
Luzin space.

If, in addition, F is Baire measurable and X is point-
Luzin, then F can be chosen such that, in addition, \( grF \setminus grf \)
is analytic.

Proof. The projection \( grF \rightarrow X \) is \( \sigma \)-dd-preserving by
[\text{FH}_1, \text{Lemma 2.5(a)}] \text{ and } [\text{FH}_1, \text{Prop. 3.1(b)}]. The graph of F is
Suslin [\text{FH}_2, \text{Prop. 4.2}], \( X \times Y \) is point-analytic [\text{FH}_2, \text{Prop.}
3.2(b)], and thus \( grF \) is point-analytic [\text{FH}_2, \text{Cor. 3.4}].

The validity of the assumptions of Theorem (b) can be
derived similarly from the extended assumptions.

Corollary 2. If \( F^{-1}:Y \rightarrow X \) is a mapping in Corollary 1,
then there exists a \( (\mathcal{G}(X) \leftarrow \text{Ba}(Y)) \)-measurable selection \( f \)
for F provided that (α) and (γ) are satisfied, and \( F^{-1} \) is
\( \sigma \)-db-preserving.

Proof. The projection \( grF \rightarrow X \) is \( \sigma \)-db-preserving by
[\text{FH}_1, \text{Lemma 2.5(b)}] \text{ and } [\text{FH}_1, \text{Prop. 3.1(b)}]. The graph of F
is point-analytic by the same arguments as in the proof of
Corollary 1.

Remark. If the assumption (β) in Corollary 1 was sup­
plied by

(β') F is \( \sigma \)-dd-preserving,

then the same assertions are valid for \( F^{-1}:Y \rightarrow X \) instead of
\( F:X \rightarrow Y \).

Similar change could be done in Corollary 2 (when \( F:X \rightarrow 
\rightarrow Y \) is a mapping).
References

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(Oblatum 28.5. 1980)

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