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REMARKS TO A MODIFICATION OF RAMSEY-TYPE THEOREMS

Martin GAVALEC, Peter VOJTÁŠ

Abstract: A typical result in the paper: if $\varkappa$ is a regular cardinal, then in any graph G of power $\geq \varkappa$ there is a subgraph H of power $\geq \varkappa$ such that every vertex of G is adjacent to precisely, none, one or $\geq \varkappa$ many of vertices of H. Similar theorems are presented for $\varkappa$ singular and for graphs describing comparability in posets and trees.

Key words: Graph, Ramsey theorem, adjacency structure, comparability graphs of posets and trees.

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1. The well-known Ramsey theorem [3] claims: "Every infinite graph contains an infinite subgraph in which either every two vertices are adjacent or no two vertices are adjacent". Recently, I. Rival and B. Sands in [4] offered a new approach to the problem: "while Ramsey's result completely describes the adjacency structure of the distinguished subgraph, it provides no information about those edges which join vertices inside the subgraph to vertices external to it". The main results in [4] are the following theorems RS 1, RS 2.

(RS 1) Every infinite graph G contains an infinite subgraph H such that every vertex of G is adjacent to precisely, none, one, or infinitely many of the vertices of H. Moreover, every vertex of H is adjacent to none or infinitely many, of the vertices of H.

In [4] an example is given which shows that, in general, the distinguished subgraph H cannot be chosen so that it is either complete or totally disconnected. However, for graphs describing the comparability in posets, Rival and Sands proved a stronger result which is closer to the Ramsey theorem.

(RS 2) Every infinite poset P of finite width contains an infinite chain C such that every element of P is comparable with none, or infinitely many, of the elements of C. Moreover, if P is countable, then C can be so chosen that every element of P is comparable with none of the elements of C or every element of a cofinite subset of C.

In this paper we consider generalizations of the above theorems for all cardinalities. Ramsey theorem with the expression "of cardinality at least $\varkappa$ " instead of "infinite" holds for weakly compact cardinals $\varkappa$ only. Such uncountable cardinals are rather large and their existence is not provable from the axioms of Zermelo-Fraenkel set theory. In contrast to this fact we show that the theorem RS 1 can be generalized, in fact, for all cardinals.

For brevity, we call a non-empty subgraph H of a graph G a $(0,1,\varkappa)$-subgraph if every vertex of G is adjacent to precisely, none, one, or at least $\varkappa$ many, of the vertices of H. Analogously - with comparability - for the notion of $(0,\varkappa)$-chain in posets.

<u>Theorem 1</u>. If G is a graph of power $\geq \varkappa$, $\varkappa$ infinite regular cardinal, then there is a $(0,1,\varkappa)$-subgraph H of G of

power $\geq \varkappa$. Moreover, every vertex of H is adjacent to none, or at least $\varkappa$ many, of the vertices of H.

If the cardinality of the graph G is a regular cardinal, then Theorem 1 gives the best possible result. For graphs of singular cardinality the situation is described by the following theorem.

<u>Theorem 2</u>. If G is a graph of power $\varkappa$, $\varkappa$ infinite singular cardinal, then for every $\alpha < \varkappa$ there is a $(0,1,\alpha)$-subgraph H of G of power $\varkappa$. Moreover, every vertex of H is adjacent to none, or at least α many of the vertices of H.

The result of Theorem 2 is the best possible. It is easy to find an example of a graph G of singular power $\varkappa$ which does not contain any $(0,1,\varkappa)$-subgraph of power $\varkappa$.

In Theorems 1,2, as well as in RS 1, the distinguished subgraph need not be complete nor totally disconnected. Even a weaker condition with the almost-completeness and almost-disconnectedness need not be satisfied (a graph H of cardinality $\varkappa$ is almost-complete if any vertex of H is adjacent to all but $< \varkappa$ vertices of H, an almost-disconnected graph is defined analogously). This follows by a "translation" of the corresponding example given in [4]: Let $\varkappa$ be an infinite cardinal number and $A = \{a_\alpha;\ \alpha \in \varkappa\}$, $B = \{b_\alpha;\ \alpha \in \varkappa\}$, $C = \{c_\alpha;\ \alpha \in \varkappa\}$ be disjoint sets of power $\varkappa$. The vertices of G we choose to be $A \cup B \cup C$. For edges of G we choose $(a_\alpha, b_\beta), (b_\alpha, c_\beta), (c_\alpha, a_\beta)$, where $\alpha, \beta \in \varkappa$ and $\alpha < \beta$. Each $(0,1,\varkappa)$-subgraph of G of power $\varkappa$ is not almost-complete nor almost-disconnected.

For graphs describing the comparability in posets a complete subgraph corresponds to a chain. Here we get closer to Ramsey, in generalizations of Theorem RS 2 for higher cardinalities. By the width $w(P)$ of a poset P we mean the least cardinal number α such that there is no antichain of cardinality α in P. A poset P is called a tree if for any $a \in P$, the set of all elements lesser than a is well-ordered.

The well known Kőnig lemma [2] implies that any infinite tree of countable width contains an infinite chain. Kőnig's methods allow also to find a chain of regular cardinality in any poset of cardinality $\varkappa$ and of countable width or in any tree of cardinality $\varkappa$ and of width $\lambda < \varkappa$ under the assumption that $2^{\nu} < \varkappa$ holds for al $\nu < \lambda$. These are not the best results, e.g. the regularity of $\varkappa$ is not necessary for trees, it suffices $2^{\nu} < cf(\varkappa)$ for all $\nu < \lambda$).

The following theorems are connected with the generalizations of Kőnig lemma as well.

Theorem 3. If P is a poset of cardinality $\varkappa$, $\varkappa$ infinite regular cardinal, $w(P) < \omega$, then there is a $(0,\varkappa)$-chain in P.

Theorem 4. If $\varkappa$ is a singular cardinal, then there is a poset P of cardinality $\varkappa$, $w(P) = 3$ such that there is no $(0,\varkappa)$-chain in P.

Thus, Theorem RS 2 cannot be generalized to singular cardinalities. The generalization to regular cardinalities involves the condition $w(P) < \omega$ which cannot be weakened even to $w(P) \leq \omega$. However, for a tree T the condition $w(T) < \varkappa$ suffices. Further weakening to $w(T) \leq \varkappa$ depends on Suslin's hypothesis (in fact, it is equivalent to it), which itself is an independent statement of Zermelo Fraenkel set theory ([1],[6],[5]).

Theorem 5. If $\varkappa$ is an infinite regular cardinal, then there is a poset P of cardinality $\varkappa$, $w(P) = \omega$, such that there is no $(0,\varkappa)$-chain in P.

Theorem 6. If T is a tree of cardinality $\varkappa$, $\varkappa$ infinite regular cardinal, $w(T) = \lambda < \varkappa$ s.t. $2^{\nu} < \varkappa$ holds for any $\nu < \lambda$. Then there is a $(0,\varkappa)$-chain in T.

Theorem 7. If $\varkappa$ is an infinite regular cardinal, then the existence of a tree T of cardinality $\varkappa$, $w(T) = \varkappa$ with no $(0,\varkappa)$-chain in T is equivalent to the existence of a Suslin's $\varkappa$ -tree, i.e. a tree S of cardinality $\varkappa$, $w(S) = \varkappa$ with no chain of cardinality $\varkappa$ in S.

The condition concerning regularity of $\varkappa$ in Theorem 6 is substantial. A trivial construction gives an example of a tree T of singular cardinality $\varkappa$, $w(T) < \varkappa$ with no $(0,\varkappa)$-chain in T.

2. In this section we give proofs of Theorems 1 - 7. We want to stress here that, what Theorem 1 concerns, the substantial work has been done in [4]. Our proof of Theorem 1 is a modification of the one in [4]. However, for the reader's convenience, we bring here the complete proof.

Let us start with some definitions. The graphs are assumed to be ordered pairs $G = (V,E)$ where edges form a binary, non-reflexive, symmetric relation E on the set of vertices V. For $H \subseteq V$ we speak about a subgraph H of G meaning the structure $(H, E \cap H^2)$.

The neighborhood of a vertex $a \in V$ is the set $N(a) = \{x \in V; (x,a) \in E\} \cup \{a\}$, for $A \subseteq V$ we set $N(A) = \bigcup \{N(a); a \in A\}$. Let $\mathcal{Q}$ be a set of cardinals, we say that $H \subseteq V$ is an $\mathcal{Q}$-subgraph of G if for any vertex $x \in V$ either

(1) $|N(x) \cap H| \in \mathcal{A}$ or

(2) $|N(x) \cap H| \geq \sup \mathcal{A}$ holds true.

If (1) holds true for all $x \in V$, we say that H is a strictly $\mathcal{A}$-subgraph of G.

Proof of Theorem 1. Let $G = (V,E)$ be a graph with $|V| \geq \varkappa$. Denote

$$F = \{x \in V; |N(x)| < \varkappa\}$$

$$T = \{x \in V; |N(x) \cap F| < \varkappa\}$$

The proof splits into three cases.

Case I. Let $|F| < \varkappa$. Put $H = V - N(F)$. By regularity of $\varkappa$, H is a $(0,\varkappa)$-subgraph of G of cardinality $\geq \varkappa$.

Case II. Let $|F| \geq \varkappa$ and for any $x \in F$ let $N(x) \subseteq T$. The set $H = \{x_\xi; \xi \in \varkappa\}$ we choose by induction in such a way that $x_\xi \in F - \bigcup\{N(N(x_\eta)); \eta \in \xi\}$ for any $\xi \in \varkappa$. Then H is a strictly (0,1)-subgraph of G. Note that in this case it is possible to take H of the same power as F.

Case III. Let $|F| \geq \varkappa$ and assume that there is an element $x \in F$ with $N(x) \not\subseteq T$. By transfinite induction through $\alpha \in \varkappa$ we choose n increasing sequence of ordinals $\{\nu_\alpha; \alpha \in \varkappa\}$ and a set of vertices of F $\{x_\xi; \xi \in \nu_\alpha\}$ as follows.

Take $x_0 \in F$ such that $N(x_0) \not\subseteq T$ and put $\nu_0 = 1$.

For $\alpha \in \varkappa$ assume that $\{\nu_\gamma; \gamma \in \alpha\}$ and $\{x_\xi; \xi < \sup\{\nu_\gamma; \gamma \in \alpha\}\}$ are already chosen. Put $\nu_\alpha^* = \sup\{\nu_\gamma; \gamma < \alpha\}$

$A_\alpha = \{y \in G - T; (\exists \xi \in \nu_\alpha^*)((y,x_\xi) \in E)\}$

$B_\alpha = \{N(y) \cap F; y \in A_\alpha\}$.

Take $\nu_\alpha = \nu_\alpha^* + |B_\alpha|$ and a numbering of B_α,

$$B_\alpha = \{C_\xi; \nu_\alpha^* \leq \xi < \nu_\alpha\}$$

For ξ such that $\nu_\alpha^* \leq \xi < \nu_\alpha$ take x_ξ such that $x_\xi \in C_\xi$ and

$x_\xi \notin \bigcup \{N(N(x_\eta)\cap T)\cap F;\ 0 \le \eta < \xi\}$. Then $H = \{x_\xi;\ \xi \in \nu_\alpha;\ \alpha \in \varkappa\}$ is a strictly $(0,1,\varkappa)$-subgraph of G. (Hint: if a vertex e is adjacent to x_ξ, x_η, $\eta < \xi < \varkappa$, then $e \in A_\alpha$ for some $\alpha \in \varkappa$. Then $N(e)\cap F \in B_\alpha$ for cofinally many $\alpha \in \varkappa$.)

Proof of Theorem 2. Assume $cf(\varkappa) = \lambda$ and let $\{\varkappa_\xi;\ \xi \in \lambda\}$ be an increasing sequence of regular cardinals greater than α such that $\varkappa = \sup\{\varkappa_\xi;\ \xi \in \lambda\}$ and $(\forall \xi \in \lambda)(\varkappa_\xi > \sup\{\varkappa_\eta;\ \eta < \xi\})$ hold true.

For $\xi \in \lambda$ denote

$F_\xi = \{a \in V;\ |N(a)| < \varkappa_\xi\}$

$T_\xi = \{a \in V;\ |N(a)\cap F_\xi| < \varkappa_\xi\}$

Case I. Assume $|F_\xi| < \varkappa_\xi$ for some $\xi \in \lambda$, then $|V - N(F_\xi)| = \varkappa$ and $V - N(F_\xi)$ is a $(0,\alpha)$-subgraph of G.

Case II. Assume $|F_\xi| \ge \varkappa_\xi$ for any $\xi \in \lambda$ and let $N(F_\xi) \subseteq T_\xi$ hold for any ξ belonging to a cofinal subset $L \subseteq \lambda$. For any $\xi \in L$ there is $H_\xi \subseteq F_\xi$, $|H_\xi| = \varkappa_\xi$ such that H_ξ is a strictly $(0,1)$-subgraph of G and $H_\eta \subseteq H_\xi$ holds true for $\eta \le \xi$ (use the proof of Theorem 1, case II). Then $H = \bigcup\{H_\xi;\ \xi \in L\}$ is a strictly $(0,1)$-subgraph of G with card $(H) = \varkappa$.

Case III. Assume $|F_\xi| \ge \varkappa_\xi$ for any $\xi \in \lambda$ and let $N(F_\xi) \nsubseteq T_\xi$ hold for any ξ belonging to a cofinal subset $L^* \subseteq \lambda$. For any $\xi \in L^*$ there is a $H_\xi^* \subseteq F_\xi$ of cardinality $\varkappa_\xi$, which is a $(0,1,\varkappa_\xi)$-subgraph of G. Moreover, for any $c \notin T_\xi$, $N(c)\cap H_\xi$ is of cardinality 0 or $\varkappa_\xi$ (use the proof of Theorem 2, case III).

Put $H_\xi = H_\xi^* - N(N(\bigcup\{H_\eta;\ \eta \in \xi\cap L^*\})\cap T_\xi)\cap F_\xi$, $H = \bigcup\{H_\xi;\ \xi \in L^*\}$. If we denote by $\mathcal{a}$ the closure (in the ordinal topology on $\varkappa$) of the set $\{0,1,\varkappa_\xi;\ \xi \in L^*\}$ then H is strictly $\mathcal{a}$-subgraph of G with $|H| = \varkappa$.

Proof of Theorem 3. Let P be a poset of regular cardinality $\varkappa$ and of finite width n. By König-type argument it is possible to show that P contains a chain C of cardinality $\varkappa$. In what follows we proceed by contradiction and assume that there are no $(0, \varkappa)$-chains in P.

For $x \in P$ denote $(x\rangle = \{y \in P; y \leq x\}$

$\langle x) = \{y \in P; x \leq y\}$ and put

$C_0 = \{x \in C; |C \cap (x\rangle| < \varkappa\}$

$C_1 = \{x \in C; |C \cap \langle x)| < \varkappa\}$

The assumption $|C_0 \cup C_1| < \varkappa$ implies that $C - (C_0 \cup C_1)$ is a $(0, \varkappa)$-chain in P. Thus, without loss of generality, we may suppose that $|C_0| = \varkappa$ and, moreover, that the ordinal type of C_0 is $\varkappa$. Then a chain K in P and a regular cardinal λ, $0 < \lambda < \varkappa$ can be found such that

(i) the ordinal type of K is $\lambda \times \varkappa$

(ii) there is no chain H in P of the ordinal type $\varkappa$ such that $(\forall x \in H)(\forall y \in K)(x \geq y)$

Claim. There exists a sequence $(K_i; i \in \omega)$ of chains in P such that each K_i fulfils (i), (ii) with the same λ and

(iii) $(\forall i \in \omega)(\forall x \in K_{i+1})(K_i \cap \langle x) = \emptyset)$

(iv) the function f defined for $x \in K_{i+1}$ by $f(x) = \min (K_i - (x\rangle)$ is an order isomorphism of K_{i+1} into K_i. (Therefore, by (i), $f(K_{i+1})$ is cofinal in K_i.)

From the claim, Theorem 3 follows. We come to contradiction by constructing an antichain $x_0 \in K_0, x_1 \in K_1, \ldots, x_{n-1} \in K_{n-1}$. The element x_{n-1} we choose arbitrarily, x_{i-1} in such a way that $x_{i-1} \not\leq x_i, \ldots, x_{n-1}$. By (i), (iv), this choice is always possible. By (iii) and by the cofinality mentioned in (iv) we have $x_{i-1} \not\geq x_i, \ldots, x_{n-1}$.

It remains to prove the claim. We set $K_0 = K$ and show how to construct K_{i+1} from K_i. By inductional assumption the ordinal type of K_i is $\lambda \times \varkappa$. For $\xi \in \lambda$ denote by $K^{(\xi)}$ the subchain of K_i, that corresponds to $\{\xi\} \times \varkappa$ in $\lambda \times \varkappa$.

Further, denote $M_i = \{x \in P; K_i \cap \langle x) = \emptyset \,\&\, K_i \neq K_i \cap (x\rangle \neq \emptyset\}$ and for $x \in M_i$ put $f(x) = \min (K_i - (x\rangle)$. The nonexistence of $(0, \varkappa)$-chains in P implies that M_i is non-empty and $f(M_i)$ is cofinal in K_i. Moreover, $f(M_i)$ must be cofinal in $K^{(\xi)}$ for any $\xi \in \lambda$. Thus, the ordinal type of $f(M_i)$ is $\lambda \times \varkappa$. The same ordinal type has any subset of K_i which is cofinal in $K^{(\xi)}$ for all ξ belonging to a cofinal subset of λ. Such subsets of K_i we shall call doubly cofinal in K_i.

By the axiom of choice it is easy to construct $M \subseteq M_i$ such that $f|M$ is a bijection of M onto $f(M) = f(M_i)$. Then $f|M$ is a bijection order homomorphism, but not an isomorphism, because M need not be a chain. Then we accomplish the last step of the proof in

Lemma. In any subset $\bar{M} \subseteq M$ such that $f(\bar{M})$ is doubly cofinal in K_i, there is a subchain $\bar{K} \subseteq \bar{M}$ of ordinal type $\lambda \times \varkappa$.

Proof of the lemma goes by induction on $w(\bar{M})$. For $w(\bar{M}) = 2$, $\bar{M}$ itself is a chain. Further we assume that the lemma holds for subsets of the width $< k = w(\bar{M})$.

By assumption, $f(\bar{M})$ is doubly cofinal in K_i, so there is a cofinal subset $L \subseteq \lambda$ such that $f(\bar{M})$ is cofinal in $K^{(\xi)}$ for any $\xi \in L$. Thus, for $\xi \in L$, the set $M^{(\xi)} = f^{-1}(K^{(\xi)}) \cap \bar{M}$ is of cardinality $\varkappa$ and, by a Ramsey-type reasoning, $M^{(\xi)}$ contains a chain of cardinality $\varkappa$. Without loss of generality we may assume that $M^{(\xi)}$ itself is a chain for $\xi \in L$ and $M^{(\xi)} = \emptyset$ for $\xi \notin L$.

For λ finite, i.e. for $\lambda = 1$, the lemma is proved. Assume that λ is infinite. Then the cofinality of $\lambda \times \varkappa$ is equal to λ. One can find a chain $\overline{K}$ in $\overline{M}$ and a regular cardinal $\overline{\lambda}$, $0 < \overline{\lambda} < \varkappa$ such that

($\overline{i}$) the ordinal type of $\overline{K}$ is $\overline{\lambda} \times \varkappa$

($\overline{ii}$) there is no chain H in $\overline{M}$ of the ordinal type $\varkappa$ such that $(\forall x \in H)(\forall y \in \overline{K})(x \geq y)$.

If $\overline{\lambda} = \lambda$, the lemma is proved. Assume $\overline{\lambda} < \lambda$, then $f(\overline{K})$ is not doubly cofinal in K_i. Denote by $\overline{L}$ the set of all upper bounds in L of the set $\{\xi \in L;\ f(\overline{K})$ is cofinal in $K^{(\xi)}\}$, then, by ($\overline{i}$) we have $f(\overline{K}) \cap K^{(\xi)} = \emptyset$ for any $\xi \in \overline{L}$.

Further denote $\overline{P} = \cup\{M^{(\xi)};\ \xi \in \overline{L}\}$. For $x \in \overline{K}$, $z \in \overline{P}$ we have either $x \parallel z$ or $x < z$, but by ($\overline{ii}$), no $z \in \overline{P}$ can fulfil $x < z$ for all $x \in \overline{K}$. Thus, denoting $P_x = \{z \in \overline{P}; x \parallel z\}$ we get $\overline{P} = \cup\{P_x;\ x \in \overline{K}\}$. For $x, y \in \overline{K}$, $x \leq y$ we have $P_x \subseteq P_y$.

If there is $x \in \overline{K}$ such that $f(P_x)$ is doubly cofinal in K_i, then, in view of $w(P_x) < k$, the inductional hypothesis gives a chain of type $\lambda \times \varkappa$ in P_x.

If $f(P_x)$ is not doubly cofinal in K_i, denote by ξ_x the least ordinal such that $f(P_x)$ is not cofinal in $K^{(\xi)}$ for any $\xi \geq \xi_x$. For $x, y \in \overline{K}$, $x \leq y$ we have $\xi_x \leq \xi_y$.

Case I. If $\overline{\lambda}$ is infinite, then the cofinality of $\overline{K}$ is $\overline{\lambda} < \lambda$. So there exists $\overline{\xi} \in \overline{L}$ such that $f(P_x)$ is not cofinal in $K^{(\xi)}$ for any $x \in \overline{K}$, $\xi \in L$, $\xi \geq \overline{\xi}$. Then $f(\overline{P}) = \cup\{f(P_x);\ x \in \overline{K}\}$ is not cofinal in $K^{(\xi)}$ for $\xi \in L$, $\xi \geq \overline{\xi}$ as well. This leads to a contradiction with double cofinality of $f(\overline{P})$ in K_i.

Case II. If $\overline{\lambda} = 1$, then the cofinality of $\overline{K}$ is $\varkappa > \lambda$. Again there exists $\overline{\xi} \in \overline{L}$ such that $f(P_x)$ is not cofinal in $K^{(\xi)}$ for any $x \in \overline{K}$, $\xi \in L$, $\xi \geq \overline{\xi}$. By the repeated König-type reasoning we can find elements $x_0 < x_1 < \dots$ in $\overline{K}$, infinite subsets $R_0 \supseteq R_1 \supseteq \dots$ of $\overline{L} \cap \langle \xi)$ and chains $\{ Z_{m\xi} \in M^{(\xi)}; \xi \in R_m \}$, $m \in \omega$ such that $x_m \leq z_{m\xi}$, $x_{m+1} \parallel Z_{m\xi}$ for any $m \in \omega$, $\xi \in R_m$. Then for $\xi_0 < \xi_1 < \dots < \xi_{n-1} \in R_{n-1}$ the elements $z_{n-1,\xi_0}; z_{n-2,\xi_1}; \dots; z_{0,\xi_{n-1}}$; form an antichain contradicting $w(P) = n$. The proof of the lemma is complete.

Proof of Theorem 4. Assume $\varkappa$ is a singular cardinal of cofinality λ, let $\{ \varkappa_\xi ; \xi \in \lambda \}$ be a sequence of lesser cardinals converging to $\varkappa$. Define a partial order on $P = \{ (\xi, \alpha), \xi \in \lambda \ \& \ \alpha \in \varkappa_\xi \}$ as follows: $(\xi, \alpha) \prec (\xi', \alpha')$ if either $\xi = \xi' \ \& \ \alpha \geq \alpha'$ or $\xi < \xi' \ \& \ (\alpha \neq \emptyset \vee \alpha' = \emptyset)$. There is no $(0, \varkappa)$-chain in $(P, \prec)$ and $w(P) = 3$.

Proof of Theorem 5. For $(P, \leq)$ we take the cartesian product $\varkappa \times \omega$ with coordinate-wise orderding. It is evident that $w(P) = \omega$ and that there is no $(0, \varkappa)$-chain in P, if $\varkappa$ is regular.

Proof of Theorem 6. Assume T is an infinite tree of regular cardinality $\varkappa$ and of width $\lambda < \varkappa$ such that $2^\nu < \varkappa$ holds for any $\nu < \lambda$. Then any chain in T contains $< \lambda$ splitting points. By the König-type argument we can prove that there is a chain C of cardinality $\varkappa$ in T. The splitting points are not cofinal in C, so leaving out an initial interval from C we get a chain of cardinality $\varkappa$ which is the $(0, \varkappa)$-chain in T.

Proof of Theorem 7 is essentially the same as the previous proof.

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