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A NOTE ON THE SPLITTING LENGTH OF A FINITE DIRECT SUM
OF MIXED ABELIAN GROUPS OF RANK ONE

Ladislav BICAN

Abstract: The purpose of this note is to show that the splitting length of a finite direct sum A of mixed abelian groups of rank one does not depend on the splitting lengths of the summands provided that the rank of A is greater than 1 and at least one of the summands is non-splitting. More precisely, it is shown that the splitting length of a direct sum of mixed abelian groups A_1, A_2, \dots, A_m of rank one with the splitting lengths $k_1 \leq k_2 \leq \dots \leq k_m$, $m \geq 2$, $k_m \geq 2$, can take an arbitrary value from the set $\{k_m, k_m+1, \dots, \infty\}$.

Key words: Splitting length, p -height sequence.

Classification: Primary 20K25

Irwin, Khabbaz and Rayna [8] have studied the splitting properties of the tensor product of mixed abelian groups. They defined the splitting length of a mixed group G as the infimum of the set of all positive integers n such that the n -th tensor $G^n = G \otimes G \otimes \dots \otimes G$ splits and they constructed a mixed group of rank one having the splitting length n for every positive integer n . In my previous paper [3] I have characterized the mixed abelian groups of rank one having the splitting length n and in [4] I have characterized all pairs A, B of mixed abelian groups of rank one having the property that the tensor product $A \otimes B$ splits. In this note we are going to prove the following result.

Theorem. Let k_1, k_2, \dots, k_m , $m \geq 2$, be arbitrary positive integers, not all equal to 1. Then for each ℓ , $\max\{k_1, k_2, \dots, k_m\} \leq \ell \leq \infty$, there exist abelian groups A_1, A_2, \dots, A_m such that each A_i has the splitting length k_i , $i=1, 2, \dots, m$, and the direct sum $A = A_1 \oplus A_2 \oplus \dots \oplus A_m$ has the splitting length ℓ .

Thus, the splitting length of a finite direct sum A of mixed abelian groups of rank one does not depend on the splitting lengths of the summands provided that the rank of A is at least 2 and at least one of the summands is non-splitting.

By the word "group" we shall always mean an additively written abelian group. As in [1], we use the notions "characteristic" and "type" in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^A(a)$, $\tau^A(a)$ and $\hat{\tau}^A(a)$ denote respectively the p -height, the characteristic and the type of the element a in the group A . π will denote the set of all primes. If T is a torsion group, then T_p is the p -primary component of T and similarly, if $\pi' \subseteq \pi$ then $T_{\pi'}$ is defined by $T_{\pi'} = \sum_{p \in \pi'}^{\oplus} T_p$. If $\pi' \subseteq \pi$ and if A is a mixed group with the torsion part $T(A)$, $T(A)_{\pi'} = 0$, then for each subset $S \subseteq A$ the symbol $\langle S \rangle_{\pi'}^A$ denotes the π' -pure closure of S in A , the existence of which is easily seen.

For a mixed group A with the torsion part $T(A)$ we denote by \bar{A} the factor-group $A/T(A)$ and for $a \in A$, \bar{a} is the element $a + T(A)$ of \bar{A} . The symbol $|a|$ means the order of the element $a \in A$. The rank of a mixed group A is that of \bar{A} . The set of all positive integers is denoted by \mathbb{N} , $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Other notation will be essentially the same as in [5].

It has been proved in [1; Theorem 2] that a mixed group A of rank one splits if and only if each element $a \in A \setminus T(A)$ has

a non-zero multiple ma such that $\hat{c}^A(ma) = \hat{c}^{\bar{A}}(\bar{a})$ and ma has a p -sequence whenever $h_p^{\bar{A}}(\bar{a}) = \infty$ (i.e. there exist elements $h_0^{(p)} = ma, h_1^{(p)}, \dots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}, n=0,1,\dots$). Recall [3], that the p -height sequence of an element $a \in A$ is the double sequence $\{k_i, l_i\}_{i=0}^{\infty}$ of elements of $\mathbb{N}_0 \cup \{\infty\}$ defined inductively in the following way: Put $k_1 = k_0 = l_0 = 0$ and $l_1 = h_p^A(a)$. If k_i, l_i are defined and either $h_p^A(p^{k_i}a) = l_i = \infty$, or $l_i < \infty$ and $h_p^A(p^{k_i+k}a) = l_i + k$ for all $k \in \mathbb{N}$ then put $k_{i+1} = k_i$ and $l_{i+1} = l_i$. If $l_i < \infty$ and there are $k \in \mathbb{N}$ with $h_p^A(p^{k_i+k}a) > l_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^A(p^{k_{i+1}}a) = l_{i+1} > l_i + k_{i+1} - k_i$.

For the sake of simplicity we shall use the notation $a^r = a \otimes a \otimes \dots \otimes a \in A^r, r \in \mathbb{N}$. Moreover, the symbols $A^r \otimes B^0$ and $a^r \otimes b^0, r \in \mathbb{N}$, will simply denote A^r and a^r , respectively.

If $\pi' \subseteq \pi$ and $l \in \mathbb{N}_0$ then we shall say that a torsion-free group A of rank one is of the type $(\pi'; l)$ if it contains an element a such that $h_p^A(a) = l$ for each $p \in \pi'$ and $h_p^A(a) = 0$ for each $p \in \pi \setminus \pi'$. Further, if $\pi' \subseteq \pi$ and $k, l, m \in \mathbb{N}_0, m > k + l$, then we shall say that a mixed group A of rank one is of the type $(\pi'; k, l, m)$ if $T(A)_{\pi \setminus \pi'} = 0$ and $A \setminus T(A)$ contains an element a such that for each prime $p \in \pi'$ the p -height sequence of a in A is $\{k_i, l_i\}_{i=0}^{\infty}$, where $k_2 = k_3 = \dots = k, l_1 = l, l_2 = l_3 = \dots = m$ and for each prime $p \in \pi \setminus \pi'$ the p -height sequence of a in A is $\{k_i, l_i\}_{i=0}^{\infty}$, where $k_0 = k_1 = \dots = l_0 = l_1 = \dots = 0$.

We start our investigations with some preliminary lemmas.

Lemma 1. If $\pi' \subseteq \pi$ and $l \in \mathbb{N}_0$ are arbitrary then there

exists a torsionfree group A of rank one and of the type $(\pi'; \ell)$.

Proof: Let $\pi' = \{p_i | i \in I\}$, $U = \langle \tilde{a} \rangle \oplus_{i \in I} \langle a_i \rangle$ be a free group and $V = \langle \tilde{a} - p_i^\ell a_i | i \in I \rangle$ be its subgroup. It is an easy exercise to show that the factor-group $A = U/V$ is torsionfree of rank one and the element $a = \tilde{a} + V$ has the desired properties.

Lemma 2. Let $\pi' \subseteq \pi$ be an arbitrary set of primes. If $k, \ell, m \in \mathbb{N}$, $m > k + \ell$, then there exist a mixed group A of rank one and of the type $(\pi'; k, \ell, m)$.

Proof: Let $\pi' = \{p_i | i \in I\}$, $U = \langle \tilde{a} \rangle \oplus_{i \in I} (\langle a_i^{(1)} \rangle \oplus \langle a_i^{(2)} \rangle)$ be a free group and $V = \langle \tilde{a} - p_i^\ell a_i^{(1)}, p_i^k \tilde{a} - p_i^m a_i^{(2)} | i \in I \rangle$ be its subgroup. Obviously, the factor-group $A = U/V$ is of rank one and we are going to show that the element $a = \tilde{a} + V$ has the desired properties.

If the equation $p^s x = p^r a$ is solvable in A then $p^r \tilde{a} = p^s (\lambda \tilde{a} + \sum_{i \in I} \lambda_i a_i^{(1)} + \sum_{i \in I} \mu_i a_i^{(2)} + \sum_{i \in I} \rho_i (\tilde{a} - p_i^\ell a_i^{(1)}) + \sum_{i \in I} \sigma_i (p_i^k \tilde{a} - p_i^m a_i^{(2)}))$ (all the sums have only a finite number of non-zero terms) and consequently

$$(1) \quad p^r = p^s \lambda + \sum_{i \in I} \rho_i + \sum_{i \in I} p_i^k \sigma_i,$$

$$(2) \quad 0 = p^s \lambda_i - p_i^\ell \rho_i, \quad i \in I,$$

$$(3) \quad 0 = p^s \mu_i - p_i^m \sigma_i, \quad i \in I.$$

If $p \notin \pi'$ then $p^s | \rho_i$, $p^s | \sigma_i$, $i \in I$, by (2) and (3), hence $p^s | p^r$ by (1) and $h_p^A(p^r a) = r$.

Assume now that $p = p_j$ for some $j \in I$. If $r \in \{0, 1, \dots, k-1\}$ then $p_j^r \tilde{a} = p_j^{r+\ell} a_j^{(1)} + p_j^r (\tilde{a} - p_j^\ell a_j^{(1)})$ and so $h_p^A(p^r a) \geq r + \ell$. If $s > r + \ell$ then (2) and (3) yield $\rho_i \equiv 0 \pmod{p^s}$, $\sigma_i \equiv 0$

$(\text{mod } p^s)$, $j \neq i \in I$, and $\rho_j \equiv 0 \pmod{p^{s-l}}$. Hence, by (1), $p^r \equiv p^k \sigma_j \pmod{p^{s-l}}$ and so $1 \equiv p^{k-r} \sigma_j \pmod{p^{s-l-r}}$, a contradiction. Thus $h_p^A(p^r a) = r + l$ for each $r \in \{0, 1, \dots, k-1\}$.

Suppose now that $r \geq k$. Then obviously $p_j^{r \wedge} = p_j^{m+r-k} s_j^{(2)+} + p_j^{r-k} (p_j^{k \wedge} - p_j^m a_j^{(2)})$ and so $h_p^A(p^r a) \geq m + r - k$. If $s > m + r - k$ then (2) and (3) yield $\rho_i \equiv 0 \pmod{p^s}$, $\sigma_i \equiv 0 \pmod{p^s}$, $j \neq i \in I$, and $\rho_j = 0 \pmod{p^{s-l}}$, $p^{s-m} \sigma'_j = \sigma_j$ for a suitable integer σ'_j . Hence, by (1), $p^r \equiv p^{s-m+k} \sigma'_j \pmod{p^{s-l}}$ and so $1 \equiv p^{s-m-r+k} \sigma'_j \pmod{p^{s-l-r}}$, a contradiction. Thus $h_p^A(p^r a) = m + r - k$ for each $r \geq k$ and the proof is complete.

Lemma 3. Let A be a mixed group of rank one. If $\pi' \subseteq \pi$ is infinite and if A is of the type $(\pi'; k-1, 1, k+1+m)$, $k \geq 2$, $m \in \mathbb{N}_0$, then A has the splitting length k and for each $r \in \{1, 2, \dots, k-1\}$ the tensor power A^r is of the type $(\pi'; k-r, r, r(m+1)+k)$.

Proof: Obviously, $(k-1)(k+1+m-(k-1))-(k-1) = (k-1)(m+1) > 0$ and A has the splitting length k by [3; Theorem].

If $a \in A \setminus T(A)$ is an element having the properties stated in the definition of the group of the type $(\pi', k-1, 1, k+1+m)$ then the assumption $T(A)_{\pi \setminus \pi'} = 0$ obviously yields $h_p^{A^r}(p^s a^r) = s$ for each prime $p \in \pi \setminus \pi'$.

Assume now that $p = p_j$ for some $j \in I$. It is easy to see that for each $a' \in A$ the p -heights of the elements $a', a' + T_{\pi \setminus \{p\}}$ in the corresponding groups are the same and from this it easily follows that we can restrict ourselves to the case of $T(A)$ p -primary. If $pa_1 = a$, $p^{k+1+m} a_2 = p^{k-1} a$ and $t = p^{m+1} a_2 - a_1$ then by [4; Lemma 8] and [3; Lemma 8] the group A decomposes into $A = \langle t \rangle \oplus V \oplus \langle a_2 \rangle_{\pi \setminus \{p\}}^A$, where $\langle t \rangle \oplus V = T(A)$. Moreover,

$a = p^{m+2}a_2 - pt$ and so $h_p^A(a^r) = r$. Finally, $p^{k-r}a^r =$
 $= p^{k-1}(a \otimes a_1^{r-1}) = p^{k+1+m}(a_2 \otimes a_1^{r-1}) = \dots = p^{(m+1)r+k}a_2^r$, from
 which the assertion follows easily.

Lemma 4. Let $\pi' \subseteq \pi$ be infinite and A, B be mixed groups of rank one and of the type $(\pi'; k-1, 1, k+m+2)$, $(\pi'; (m-1)(\ell-1), \ell-1, m\ell)$, $k \geq 2$, $\ell \geq m \geq 3$, respectively. Then $A^{\ell-2} \otimes B$ does not split. The same holds if A is a torsionfree group of rank one and of the type $(\pi'; m-1)$.

Proof: If $\ell - 2 < k$ then $A^{\ell-2}$ is of the type $(\pi'; k-\ell + 2, \ell-2, (\ell-2)(m-2)+k)$ by Lemma 3 and if $\ell - 2 \geq k$ then $A^{\ell-2}$ splits and its torsionfree direct summand is of the type $(\pi'; (m-1)(\ell-2))$. In both cases we have $(\ell-2)(m-2)+k - (k-\ell+2) - (m-1)(\ell-1) = (m-1)(\ell-2) - (m-1)(\ell-1) = -(m-1) < 0$ and $A^{\ell-2} \otimes B$ does not split by [4; Theorem] (or [4; Corollary 3]). The rest is similar.

Lemma 5. Let A_1, A_2, \dots, A_m be mixed abelian groups, $m \in \mathbb{N}$. Then $(A_1 \oplus A_2 \oplus \dots \oplus A_m)^{\ell}$ splits if and only if $A_1^{r_1} \otimes A_2^{r_2} \otimes \dots \otimes A_m^{r_m}$ splits for all $r_1, r_2, \dots, r_m \in \mathbb{N}_0$ with $\sum_{i=1}^m r_i = \ell$.

Proof: The assertion follows easily from the simple fact that $A \otimes B$ splits if and only if both A and B are splitting.

Proof of Theorem: With respect to Lemma 5 we can suppose that $k_1 \leq k_2 \leq \dots \leq k_m < \infty$. Now we shall divide the proof into several cases.

I. $\ell < \infty$.

1. Let $k_m \geq 3$ and let $j \in \{0, 1, \dots, m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} \leq \dots \leq k_m$. For each $i \in \{1, 2, \dots, j\}$ let A_i be a torsionfree group of rank one of the type $(\pi'; k_m-1)$, for each $i \in \{j+1, \dots, m-1\}$ let A_i be a mixed group of rank one

of the type $(\pi; k_i - 1, 1, k_i + k_m - 2)$ and let A_m be a mixed group of rank one of the type $(\pi; (k_m - 1)(\ell - 1), \ell - 1, k_m \ell)$. The groups $A_i, i=1, 2, \dots, m$, have the splitting length k_i by Lemma 3, the group $A_{m-1}^{\ell-2} \otimes A_m$ does not split by Lemma 4, so that with respect to Lemma 5 it remains to show that $A_1^{r_1} \otimes A_2^{r_2} \otimes \dots \otimes A_m^{r_m}$ splits whenever $r_1, r_2, \dots, r_m \in \mathbb{N}_0$ and $\sum_{i=1}^m r_i = \ell$. By [1; Theorem 2] it suffices to show that

$$h_p^{r_1 \otimes \dots \otimes r_m} (a_1^{r_1} \otimes a_2^{r_2} \otimes \dots \otimes a_m^{r_m}) = \ell(k_m + r_m - 1) (a_1, a_2, \dots, a_m)$$

are the elements having the properties stated in the definition of the groups of the corresponding types). For each $p \in \pi$ we have $p^{k_m - 1} a_i^{(1)} \in a_i, a_i^{(1)} \in A_i, i \in \{1, 2, \dots, j\}, p a_i^{(1)} = a_i,$

$$p^{k_i + k_m - 2} a_i^{(2)} = p^{k_i - 1} a_i, a_i^{(1)}, a_i^{(2)} \in A_i, i \in \{j+1, \dots, m-1\}, \text{ and}$$

$$p^{\ell-1} a_m^{(1)} = a_m, p^{k_m \ell} a_m^{(2)} = p^{(k_m - 1)(\ell - 1)} a_m, a_m^{(1)}, a_m^{(2)} \in A_m. \text{ Now let}$$

k be the first integer with $r_k > 0$. If $k = m$ then $a_m^\ell = p^{\ell(\ell-1)}$

$$(a_m^{(1)})^\ell, p^{k_m(\ell-1)} a_m^{(1)} = p^{k_m \ell} a_m^{(2)} \text{ and the induction yields } a_m^\ell =$$

$$= p^{\ell(\ell + k_m - 1)} (a_m^{(2)})^\ell \text{ owing to the fact that } \ell(\ell - 1) - k_m(\ell - 1) =$$

$$= (\ell - 1)(\ell - k_m) \geq 0. \text{ Now if } k \in \{j+1, \dots, m-1\} \text{ then } p^{k_i} a_i^{(1)} =$$

$$= p^{k_i + k_m - 2} a_i^{(2)}, i \in \{k, k+1, \dots, m-1\}. \text{ If we put } \alpha = \ell + r_m(\ell - 2)$$

and $\beta = \alpha + (k_m - 2)(\ell - r_m)$ then $\alpha \geq \ell \geq k_i, i \in \{k, k+1, \dots, m-1\},$

$$\text{and } a_k^{r_k} \otimes \dots \otimes a_m^{r_m} = p^\alpha ((a_k^{(1)})^{r_k} \otimes \dots \otimes (a_m^{(1)})^{r_m}) =$$

$$= p^\beta ((a_k^{(2)})^{r_k} \otimes \dots \otimes (a_{m-1}^{(2)})^{r_{m-1}} \otimes (a_m^{(1)})^{r_m}). \text{ For } r_m = 0 \text{ we}$$

are ready and for $r_m > 0$ the inequality $\beta - k_m(\ell - 1) =$

$$= (\ell - k_m)(r_m - 1) \geq 0 \text{ yields } a_k^{r_k} \otimes \dots \otimes a_m^{r_m} = p^{\ell(k_m + r_m - 1)}$$

$$((a_k^{(2)})^{r_k} \otimes \dots \otimes (a_m^{(2)})^{r_m}). \text{ Finally, if } k \in \{1, 2, \dots, j\} \text{ then}$$

$\alpha = \ell + r_m(\ell - 2) + (k_m - 2) \sum_{i=j+1}^m r_i \geq \ell \geq k_i, i \in \{j+1, \dots, m-1\},$
 $\beta = \alpha + (k_m - 2) \sum_{i=j+1}^{m-1} r_i, \beta - k_m(\ell - 1) = (\ell - k_m)(r_m - 1)$ and the
 assertion follows as in the preceding case.

2. $k_m = 2.$

a) Let $\ell \geq 3$ and let $j \in \{0, 1, \dots, m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} = \dots = k_m = 2.$ For each $i \in \{1, 2, \dots, j\}$ let A_i be a torsionfree group of rank one of the type $(\pi; 2);$ for each $i \in \{j+1, \dots, m-1\}$ let A_i be a mixed group of rank one of the type $(\pi; 1, 1, 3)$ and let A_m be a mixed group of rank one of the type $(\pi; 2(\ell - 1), \ell - 1, 4(\ell - 1)).$ The groups $A_i, i=1, 2, \dots, m,$ have the splitting length k_i by Lemma 3 and the group $A_{m-1}^{\ell-2} \otimes A_m$ does not split by Lemma 4. The proof of the splitting of A is similar to that in 1.

b) Let $\ell = 2$ and let $j \in \{0, 1, \dots, m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} = \dots = k_m = 2.$ For each $i \in \{1, 2, \dots, j\}$ let A_i be a torsionfree group of rank one of the type $(\pi; 1)$ and for each $i \in \{j+1, \dots, m\}$ let A_i be a mixed group of rank one of the type $(\pi; 1, 1, 3).$ The groups $A_i, i=1, 2, \dots, m$ have the splitting length k_i and the splitting length of A is obviously 2.

II. $\ell = \infty.$

Let p be a prime and $j \in \{0, 1, \dots, m-1\}$ be such that $1 = k_1 = k_2 = \dots = k_j < k_{j+1} \leq \dots \leq k_m.$ For each $i \in \{1, 2, \dots, j\}$ let $A_i = \mathbb{Z}$ (the group of integers), for each $i \in \{j+1, \dots, m-1\}$ let A_i be a mixed group of rank one of the type $(\pi \setminus \{p\}; k_i - 1, 1, k_i + 1)$ and let A_m be the group generated by the elements a_0, a_1, \dots with respect to the relations $p \binom{k_m - 1}{i} a_i = \binom{k_m - 2}{i} a_0.$ The groups $A_i, i=1, 2, \dots, m-1,$ have the split-

ting length k_1 by Lemma 3 and the group A_m has the splitting length k_m by [3; Example] (see also [8]). However, for each $\ell > 1$ the group $A_{m-1}^{\ell-1}$ is p -reduced, no non-zero element from A_m has a p -sequence and hence the group $A_{m-1}^{\ell-1} \otimes A_m$ does not split by [4; Theorem]. Thus the group A is of infinite splitting length and the proof is complete.

R e f e r e n c e s

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