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Abstract: Call an abelian group separable if each finite subset is contained in a completely decomposable direct summand. We characterize mixed separable groups in terms of their torsion and torsion-free parts and a regularity condition on the embedding of the torsion part. We investigate this regularity condition in some detail and generalize to the mixed case results known in the torsion and torsion-free cases.

Key words and phrases: Mixed group, separable, completely decomposable, Ulm matrix, quasi-isomorphism.

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All groups considered in this paper are additively written abelian groups. Such a group is said to be of rank one if it is isomorphic either to a subgroup of the rationals or to a subgroup of a quasi-cyclic group $Z(p^{\infty})$ for some prime p. We call a group completely decomposable if it is a direct sum of rank one subgroups. Although the term had previously been applied only to torsion-free groups, Fuchs [1] suggested that an arbitrary abelian group G be called separable if each finite subset of G can be embedded in a completely decomposable direct summand of G. It is easily seen that a group is separable if and only if its reduced part is separable and that a reduced torsion group G is separable if and only if the subgroup $\bigcap_{n=1}^{\infty} nG$ is trivial. Hence, the theory of separable torsion groups is essentially coextensive with that of primary groups without elements of infinite height. Although torsion-free and torsion separable groups have received considerable attention, the mixed case has apparently not been dealt with heretofore. This is somewhat surprising since as we shall see the basic results on mixed separable groups are not difficult to obtain. Indeed we shall advance the theory of separable mixed groups to roughly the same state enjoyed by the torsion and torsion-free cases. It must be admitted, however, that this state is not altogether satisfactory. For example, apart from some isolated results for homogeneous groups, not a great deal is known about the structure of separable torsion-free groups beyond the facts that their direct summands are once again separable and that the countable ones are completely decomposable.

1. Separable groups. The maximal torsion subgroup of G will be denoted as tG. Recall that G is said to be mixed if $0 \neq tG \neq G$ and we say that G splits if tG is a direct summand of G. The following simple lemma is of frequent use.

Lemma 1.1. Supose that tG is separable. Then the mixed group G is separable if and only if for each finite subset X of G there is a torsion-free, completely decomposable direct summand H of G such that $X \subseteq tG + H$.

Proof. The necessity of the condition follows from the observation that every completely decomposable group splits.

Suppose $X = \{x_1, \dots, x_n\}$ and $G = H \oplus K$ where H satisfies the conditions of the lemma. Then we can write $x_i = t_1 + h_i$ where $t_i \in tG$ and $h_i \in H$ for $i = 1,2,\dots,n$. Since tG is separable, it contains a finite rank summand A which contains $\{t_1, \dots, t_n\}$. Now $tG \subseteq K$ since H is torsion-free and A is necessarily a direct summand of K-bounded pure subgroups are direct summands. Thus $A \oplus H$ is a completely decomposable direct summand of G which contains X.

In order to see that there is genuinely something new in the theory of mixed separable groups, we shall now establish a result which leads to the existence of separable groups which do not split.

<u>Proposition 1.2.</u> If tG is separable and if G/tG is an $\aleph_1$-free separable group, then G is separable.

Proof. Let $X = \{x_1, \dots, x_n\}$ be a finite subset of G. Since G/tG is separable, we have a direct decomposition $G/tG = A/tG \oplus B/tG$ where A/tG has finite rank and contains each of the cosets $x_1 + tG, \dots, x_n + tG$. But A/tG is a free group since it is countable and G/tG is $\aleph_1$-free. Thus we have a direct decomposition $A = tG \oplus H$ which yields $G = H \oplus B$. The separability of G then follows from Lemma 1.1,

<u>Corollary 1.3.</u> There exist mixed separable groups which do not split.

Proof. Let P be the product of countably many copies of the integers. Then a construction by Griffith [3] yields a group G such that tG is a direct sum of cyclic groups, $G/tG \cong P$ and every torsion-free subgroup of G is free. Since P is $\aleph_1$-free and separable, but not a free group, the group G has the desired property.

We shall see below, however, that a separable group G with G/tG countable does split. Now since a direct decomposition of G induces a decomposition of tG and of G/tG, it is easily seen that if G is separable, then both tG and G/tG are separable. The converse, of course, fails. For example, if G/tG is of rank one and if G is separable, then it is obvious from 1.1 that G must aplit. But there exist non-splitting groups G such that $G/tg \cong Q$ and tG is a direct sum of cyclic groups. We shall see, though, that if the embedding of tG in G is sufficiently nice, then G will be separable provided tG and G/tG are.

We let $h_G^p(x)$ denote the <u>height</u> in G of x at the prime p, that is, $h_G^p(x) = \alpha$ if α is the first ordinal such that $x \notin p^{\alpha+1}G$ and $h_G^p(x) = \infty$ if $x \in p^{\alpha}G$ for all ordinals α. With each $x \in G$ we associate its <u>characteristic</u> $\chi_G(x) = (\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots)$ where α_i is the height of x at the i^{th} prime. We shall say that the maximal torsion subgroup tG is <u>balanced</u> in G provided each coset x+tG contains an element a such that $\chi_{G/tG}(x+tG) = \chi_G(a)$. In the second half of this paper we shall look at this concept in greater detail, but for the moment we require only the crucial fact that G will split provided tG is balanced in G and G/tG is completely decomposable [7]. In the case when tG is separable, this was already known to Iyapin [4].

<u>Theorem 1.4.</u> The mixed group G is separable if and only both tG and G/tG are separable and tG is balanced in G.

Proof. Suppose tG is balanced in G and that both tG and

G/tG are separable. Let $X = \{x_1,\dots,x_n\}$ be any finite subset of G. As in the proof of Proposition 1.2, we have a direct decomposition $G/tG = A/tG \oplus B/tG$ with A/tG a completely decomposable group containing $x+tG,\dots,x_n+tG$. Then $tA=tG$ is balanced in A and hence $A=tG\oplus H$ by Lyapin's theorem. The desired conclusion follows from Lemma 1.1 since now $G=H\oplus B$ and $X \subseteq tG + H$.

Conversely, suppose G is separable and consider any coset $x+tG$. Then we have a direct decomposition $G = A \oplus B \oplus K$ where A is a finite rank torsion-free subgroup and B is a finite rank torsion subgroup such that x is contained in $A\oplus B$. Write $x = a + b$ where $a \in A$ and $b \in B$ and observe that $a \in x + tG$. Since $A + tG/tG$ is a direct summand of G/tG, the characteristic of $x + tG$ as computed in $A + tG/tG$ is the same as when computed in G/tG. Moreover, since $x + tG = a + tG$ is the image of a under the canonical isomorphism of A onto $A + tG/tG$, we conclude that $\chi_G(a) = \chi_A(a) = \chi_{G/tG}(x+tG)$ as desired.

<u>Corollary 1.5</u>. If G is a separable mixed group with G/tG countable, then G splits.

Proof. A countable, separable torsion-free group is completely decomposable [1, Theorem 87.1].

Since a countable primary group without elements of infinite height is necessarily a direct sum of cyclic groups, we have

<u>Corollary 1.6.</u> A countable separable group is completely decomposable.

We conclude this section by showing that direct summands of separable groups are themselves separable groups. For torsion groups this is quite trivial; whereas the difficult torsion-free case has been handled by Fuchs [2]. These special cases coupled with our Theorem 1.4 yield the general result rather easily.

Theorem 1.7. If H is a direct summand of the separable group G, then H is separable.

Proof. Write $G = H \oplus K$. Since tH is a direct summand of tG and H/tH is isomorphic to a direct summand of G/tG, it suffices to verify that tH is a balanced subgroup of H. Suppose $x \in H$. Since tG is balanced in G, there is an $a \in x + tG$ such that $\chi_G(a) = \chi_{G/tG}(x+tG)$. Notice first that $\chi_{G/tG}(x+tG) = \chi_{H/tH}(x+tH)$ since H + tG/tG is a direct summand of G/tG and x + tG is the image of x + tH under the canonical isomorphism of H/tH onto H + tG/tG. Now write $a = h+k = x+t$ where $h \in H$, $k \in K$ and $t \in tG$. Since $tG = tH \oplus tK$, $k \in tK$ and hence $h \in x + tG$. Thus $h \in x + tH$ and $h^p_{H/tH}(x+tH) = h^p_G(a) = \min\{h^p_G(h), h^p_G(k)\} \leq h^p_G(h) = h^p_H(h) \leq h^p_{H/tH}(x+tH)$ for all primes p, that is, $\chi_H(h) = \chi_{H/tH}(x+tH)$.

2. Balanced torsion subgroups. We introduce an equivalence relation on sequences of ordinals and symbols ∞ as follows: $(\alpha_1, \ldots, \alpha_n, \ldots) \sim (\beta_1, \ldots, \beta_n, \ldots)$ if and only if $\alpha_n \neq \beta_n$ for at most finitely many values of n and $\alpha_n = \beta_n$ when either is infinite. The equivalence class determined by $\chi_G(x)$ is called the type of x when G is torsion-free and G is said to be homogeneous provided all of its nonzero elements have the same type. The Ulm matrix $U_G(x)$ is a more sensitive indicator than $\chi_G(x)$ for describing how x is em-

bedded in G. It is defined as the doubly infinite matrix having as its i^{th} row the heights at the i^{th} prime p of the sequence of elements $x, px, \dots, p^n x, \dots$. The Ulm matrix plays a prominent role in the structure of countable mixed groups G with G/tG of rank one (see § 104 of [1]). The basic fact we require here is that such a group G will split if and only if it contains an element x of infinite order such that $U_G(x)$ satisfies the following three conditions:

(1) Almost all rows of $U_G(x)$ are free of gaps.

(2) No row of $U_G(x)$ has infinitely many gaps.

(3) If a row of $U_G(x)$ contains a nonfinite entry, then it contains an

Saying that the sequence $(\alpha_0, \alpha_1, \dots, \alpha_n, \dots)$ has a gap at α_n means that $\alpha_{n+1} > \alpha_n + 1$. Actually the countability assumption on G can be dropped as we shall see in 2.1 below. Let us call the Ulm matrix $U_G(x)$ <u>tame</u> if it satisfies conditions (1),(2) and (3) above. It is easily seen that an element of finite order has a tame Ulm matrix and that if x and y are elements of infinite order such that nx = my for nonzero integers n and m, then $U_G(x)$ is tame if and only if $U_G(y)$ is tame.

Proposition 2.1. Let G be a mixed abelian group with G/tG of rank one. Then the following conditions are equivalent:

(i) G splits.

(ii) tG is a balanced subgroup of G.

(iii) Each element of G has a tame Ulm matrix.

(iv) There exists some $x \in G$ of infinite order such that $\chi_G(x) = \chi_{G/tG}(x+tG)$.

(v) There exists some $x \in G$ of infinite order such that $\chi_G(x) \sim \chi_{G/tG}(x+tG)$.

Proof. The series of implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) is obvious. To show that (v) implies (iii) it suffices to show that the element x of infinite order with $\chi_G(x) \sim \chi_{G/tG}(x+tG)$ necessarily has a tame matrix. Since G/tG is torsion-free, we have for all primes p $h_G^p(x) + i \leq h_G^p(p^i x) \leq h_{G/tG}^p(p^i x+tG) = h_{G/tG}^p(x+tG) + i$. By (v) we must therefore have for almost all p the equality $h_G^p(p^i x) = h_G^p(x) + i$ for all i, that is, $U_G(x)$ satisfies condition (1) for tameness. Condition (3) for the tameness of $U_G(x)$ follows from our definition of $\sim$ and the fact that the only infinite entries in $\chi_{G/tG}(x+tG)$ are ∞ since G/tG is torsion free. Suppose finally that $U_G(x)$ fails to satisfy condition (2), that is, there is some prime p such that corresponding row of $U_G(x)$ has infinitely many gaps. Let $n = h_{G/tG}^p(x+tG)$ and choose k such that $h_G^p(p^k x) > k+n$. This contradicts the fact that $h_G^p(p^k x) \leq h_{G/tG}^p(p^k x+tG) = k + h_{G/tG}^p(x+tG) = k + n$.

As noted above (iii) implies (i) when G is countable. Assume (iii) and choose an element $x \in G$ having infinite order with tame Ulm matrix $U_G(x)$. By replacing x by a multiple of itself, we may assume that the only infinite entries in $U_G(x)$ are of the form ∞ . Therefore by Theorem 2 in [5] there is a countable group H containing an element y such that $U_H(y) = U_G(x)$ and such that the Ulm invariants of H are dominated by those of G. Then by 104.2 in [1] and a standard argument, there exists a height preserving monomorphism of H into G mapping y to x. In other words we may assume that H is a pure subgroup of G containing x with $U_H(x) = U_G(x)$. But then $G \subseteq H + tG$ because G/tG is a rank one group. Since H is countable, (iii) yields a direct decomposition $H = tH \oplus A$, from which it immediately follows that $G = tG \oplus A$.

The real utility of Proposition 2.1 seems to lie not so much in the weakening of condition (ii) to condition (v), but rather in the replacement of (ii) by the quite different sort of condition (iii). The significance of this being that $U_G(x)$ is computed within G and one no longer needs to compare elements in G/tG with corresponding elements in G. In this spirit, we state

Corollary 2.2. The group G has a balanced maximal torsion subgroup tG if and only if the Ulm matrix of each element of G is tame.

Proof. Consider any nonzero element x + tG of G/tG. Then there is a unique pure subgroup A of G such that A/tG is a rank one subgroup of G/tG containing x+tG. Moreover, all heights computed in A are the same as when computed in G since G/A is torsion-free. We then need only observe that $\chi_{G/tG}(x+tG) = \chi_{A/tG}(x+tG)$, $U_G(x) = U_A(x)$ and $\chi_G(a) = \chi_A(a)$ for any $a \in x + tG$.

Another easy consequence of 2.1 is

Corollary 2.3. The mixed group G has a balanced maximal torsiom subgroup if and only if each subgroup A with A/tG of rank one necessarily splits.

Proposition 2.4. If G/H is bounded, then tG is balanced in G if and only if tH is balanced in H.

Proof. We need only consider the special case where

$pG \subseteq H$ for some prime p. Now let z be any element of H. If $z = ny$ for some $y \in G$ and $(n,p) = 1$, then $y \in H$ and consequently $h_G^q(z) = h_H^q(z)$ for all primes $q \neq p$. The condition $pG \subseteq H$ implies that $p^\omega G = p^\omega H$ and therefore $h_G^p(z) = h_H^p(z)$ if either is infinite. On the other hand, if $h_G^p(z)$ is finite we must have $h_H^p(z) \leq h_G^p(z) \leq h_H^p(z) + 1$. These observations lead readily to the conclusion that for all $x \in H$ $U_G(x)$ is tame if and only if $U_H(x)$ is tame.

Recall that groups G and H are said to be quasi-isomorphic if there exist subgroups A and B of G and H respectively such that $A \cong B$ and both G/A and H/B are bounded. In particular, if G/H is bound then G and H are quasi-isomorphic.

Corollary 2.5. If G and H are quasi-isomorphic, then tG is balanced in G if and only if tH is balanced in H.

Corollary 2.6. If G is quasi-isomorphic to a separable group and if G/tG is separable, then G is separable.

Proof. The hypothesis implies that G contains a separable subgroup H with G/H bounded. Then tG is balanced in G by 1.4 and 2.4 and moreover tG/tH is bounded. This latter condition implies that the torsion group tG is also separable since we have $p^\omega(tG) = p^\omega(tK)$ for all primes p. We need only 1.4 again.

A group G is said to be quasi-splitting if it is quasi-isomorphic to a group which splits.

Corollary 2.7. If G is quasi-splitting, then tG is balanced in G.

In [7], C. Walker calls a coset x + H regular in G/H provided for each rank one subgroup A/H of G/H which contains x+H there is an $a \in x+H$ having the same order as x such that $\chi_A(x) = \chi_{A/H}(x+H)$. If every coset in G/H is regular, she calls H a <u>regular subgroup</u> of G. In case G/H is torsion-free, H is a regular subgroup of G if and only if each coset x+H contains an element a for which $\chi_G(a) = \chi_{G/H}(x+H)$. In particular, tG is a regular subgroup of G if and only if it is balanced in our terminology. Furthermore it is shown in [7] that the short exact sequence $A \overset{\alpha}{\rightarrowtail} G \twoheadrightarrow C$ with αA a regular subgroup of G form a proper class and hence generate a relative homological algebra. Thus the subset of Ext(C,A) determined by these regular extensions form a subgroup which we shall denote as Bext(C,A). Walker shows that the projectives for this relative homological algebra are just the completely decomposable groups, that is, Bext(C,A) = 0 for all A if and only if C is completely decomposable. The remainder of this paper is devoted to observations concerning the group Bext(F,T) when T is torsion and F is torsion-free, that is, to the group of extensions $T \rightarrowtail G \twoheadrightarrow F$ with $tG \cong T$ balanced in G.

<u>Proposition 2.8.</u> Bext(F,T) is a subgroup of Ext(F,T) which contains the maximal torsion subgroup tExt(F,T).

Proof. As established in [6], tExt(F,T) consists of equivalence classes of extensions $T \rightarrowtail G \twoheadrightarrow F$ which are quasi-splitting and hence we need only apply 2.7.

<u>Proposition 2.9.</u> Bext(F,T) = Ext(F,T) for all torsion groups T if and only if F is homogeneous of type (0,0,...,0,...).

Proof. If F is not homogeneous of type $(0,0,\dots,0,\dots)$, then it contains a rank one pure subgroup C which is not cyclic. Therefore by [5], there is a mixed group G such that $G/tG \cong C$ and G does not split. Hence tG is not balanced in G and $tG \rightarrowtail G \twoheadrightarrow C$ represents an element of Ext(C,tG) which is not in Bext(C,tG). Now the canonical epimorphism $\mathrm{Ext}(F,tG) \to \mathrm{Ext}(C,tG)$ makes Bext(F,tG) into $\mathrm{Bext}(C,tG) \neq \mathrm{Ext}(C,tG)$ and thus we cannot have Ext(F,tG) equal to Bext(F,tG). Conversely, if F is homogeneous of type $(0,0,\dots,0,\dots)$, then given any group G with $G/tG \cong F$, each rank one subgroup A/tG is free and therefore A splits. Thus tG is balanced in G by 2.3.

Corollary 2.10. Let F be a separable torsion-free group. Then Ext(F,T) = Bext(F,T) for all separable torsion groups T if and only if F is $\aleph_1$-free.

It is natural to look at the analog of Baer's problem [3] in the present context. Thus we ask whether F is necessarily completely decomposable if Bext(F,T) = 0 for all torsion groups T. The answer is negative. Call a torsion-free group F almost completely decomposable if it contains a completely decomposable subgroup C such that F/C is bounded. From constructions in §88 of [1] it follows that there exist indecomposable, almost completely decomposable groups of any rank not exceeding $2^{\aleph_0}$.

Proposition 2.11. If F is countable and almost completely decomposable, then Bext(F,T) = 0 for all torsion groups T.

Proof. Suppose $G/tG \cong F$ and tG is balanced in G. If C is completely decomposable and F/C is bounded, then G contains a subgroup H such that $H/tG \cong C$ and G/H is bounded. But then $tH = tG$ is balanced in H by 2.3 and therefore H splits, that is, G is quasi-splitting. Thus we see that $\mathrm{Bext}(F,T) = t\mathrm{Ext}(F,t)$ whenever F is almost completely decomposable. But on the other hand, $\mathrm{Ext}(F,T)$ is torsion-free whenever F is a countable torsion-free group and T is torsion (see 102.3 in [1]).

A related question, however, remains unanswered. If F is a separable torsion-free group with $\mathrm{Bext}(F,T) = 0$ for all separable torsion groups T, is F completely decomposable? It is apparently unknown whether a separable, almost completely decomposable group is necessarily completely decomposable.

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