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CORRECTION to „Extensions of the Shannon entropy to semimetrized measure spaces"

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(1) The note mentioned in the title (Comment. Math. Univ. Carolinae 21(1980), 171-192; quoted as ES in the sequel) contains an error due to which (i) two minor assertions (in ES 1.16, 3.7) are incorrect, (ii) the definition of a subentropy (ES 2.1), although correct, is not adequate (to be precise, it is too broad). The error consists in choosing an inappropriate equivalence relation on $\{WM\}$. If the relation is replaced as in (5) below, all the statements and proofs remain valid with the exception of ES 1.16, 3.7, the correct version of which is stated in (10),(11).

(2) Notation. If $\langle Q,\rho,\mu\rangle$ is a WM-space, then $\mathcal{M}(P)$ denotes the set of all measures μ' on Q such that dom μ' = dom μ, $\mu' \leq \mu$.

(3) Definition. Let $P = \langle Q,\rho,\mu\rangle$, $S = \langle T,\nu,\sigma\rangle$ be WM-spaces. Let $F \subset \mathcal{M}(P) \times \mathcal{M}(S)$ satisfy the following conditions (for convenience, we write $\mu' \sim \nu'$ instead of $\langle \mu', \nu'\rangle \in F$):

(a) $F(\mathcal{M}(P)) = \mathcal{M}(S)$, $F^{-1}(\mathcal{M}(S)) = \mathcal{M}(P)$;

(b) if $\mu_i \sim \nu_i$, $i = 1,\dots,n$, $\sum_{i=1}^{m} \mu_i \in \mathcal{M}(P)$, and $\sum_{i=1}^{m} \nu_i \in \mathcal{M}(S)$, then $\sum \mu_i \sim \sum \nu_i$; if $\mu_1 \sim \nu_1$, $a \geqq 0$,

$a\mu_1 \in \mathcal{M}(P)$, $a\nu_1 \in \mathcal{M}(S)$, then $a\mu_1 \sim a\nu_1$;

(c) $\mu \sim \nu'$ iff $\nu' = \nu$; $\mu' \sim \nu$ iff $\mu' = \mu$;

(d) if $\mu_i \in \mathcal{M}(P)$, i = 0,...,n, $\mu_0 = \sum_{i=1}^{n} \mu_i$, $\mu_0 \sim \nu_0$, then there exist $\nu_i \in \mathcal{M}(S)$ such that $\mu_i \sim \nu_i$, i = 1,...,n, and $\sum_{i=1}^{n} \nu_i = \nu_0$; if $\nu_i \in \mathcal{M}(S)$, i = 0,...,n, $\nu_0 = \sum_{i=1}^{n} \nu_i$, $\mu_0 \sim \nu_0$, there exist $\mu_i \in \mathcal{M}(P)$ such that $\mu_i \sim \nu_i$, i = 1, ...,n, and $\sum_{i=1}^{n} \mu_i = \mu_0$;

(e) if $\mu' \sim \nu'$, then $\mu' Q = \nu' T$;

(f) if $\mu_i \sim \nu_i$, then $\hat{r}(\mu_1, \mu_2) = \hat{r}(\nu_1, \nu_2)$, where $\hat{r}(\mu_1, \mu_2)$ ($\hat{r}(\nu_1, \nu_2)$, resp.) stands for $\hat{r}(\langle Q, \rho, \mu_1 \rangle, \langle Q, \rho, \mu_2 \rangle)$ ($\hat{r}(\langle T, \sigma, \nu_1 \rangle, \langle T, \sigma, \nu_2 \rangle)$, resp.), as defined in ES 1.11.

Then $\langle F, P, S \rangle$, also denoted by $F: P \longrightarrow S$, is called a conservative measure-correspondence (from P to S).

<u>Remark.</u> The definition can be simplified. E.g., (c) can be omitted, and (a) can be replaced by (a$'$) $\langle \mu, \nu \rangle \in F$. However, we prefer a detailed formulation.

(4) <u>Proposition</u>. Let $F: P \longrightarrow S$, $G: S \longrightarrow U$ be conservative measure-correspondences. Let $G * F$ consist of all $\langle \mu', \lambda' \rangle \in \mathcal{M}(P) \times \mathcal{M}(U)$ such that, for some $\nu' \in \mathcal{M}(S)$ and some $a > 0$, we have $\langle a\mu', \nu' \rangle \in F$, $\langle \nu', a\lambda' \rangle \in G$. Then $\langle G * F, P, U \rangle$ is a conservative measure-correspondence.

<u>Proof.</u> I. Put $P = \langle Q, \rho, \mu \rangle$, $S = \langle T, \sigma, \nu \rangle$, $U = \langle V, \varphi, \lambda \rangle$. Put $\Phi = G * F$. It is easy to see that $\Phi \subset \mathcal{M}(P) \times \mathcal{M}(U)$ satisfies (3a) and (3e). Clearly, $\langle \mu, \lambda \rangle \in \Phi$. Hence, if $\langle \mu, \lambda' \rangle \in \Phi$, then $\lambda' V = \mu Q = \lambda V$, which implies $\lambda' = \lambda$. Thus Φ satisfies (3c).

II. Let $\langle \mu_i, \lambda_i \rangle \in \Phi$, i = 1,...,n; let $\sum_{i=1}^{n} \mu_i \in \mathcal{M}(P)$, $\sum_{i=1}^{n} \lambda_i \in \mathcal{M}(U)$. Then there exist $\nu_i \in \mathcal{M}(S)$ and $a_i > 0$ such that $\langle a_i \mu_i, \nu_i \rangle \in F$, $\langle \nu_i, a_i \lambda_i \rangle \in G$, i = 1,...

$\ldots,n$. Choose $a>0$ such that $a<1$, $a<a_i/n$, $i = 1,\ldots,n$. Put $\nu_i' = (a/a_i)\cdot\nu_i$. Then $\langle a\mu_i , \nu_i'\rangle \in F$, $\langle \nu_i', a\lambda_i\rangle \in G$, $i = 1,\ldots,n$. Since $a<1$, $\sum_{i=1}^{n}\nu_i' \leqq \frac{1}{n}\sum_{i=1}^{n}\nu_i$, we get $\sum_{i=1}^{n} a\mu_i \in \mathcal{M}(P)$, $\sum_{i=1}^{n}\nu_i' \in \mathcal{M}(S)$, $\sum_{i=1}^{n} a\lambda_i \in \mathcal{M}(U)$. Hence, $\langle a\sum_{i=1}^{n}\mu_i , \sum_{i=1}^{n}\nu_i'\rangle \in F$, $\langle \sum_{i=1}^{n}\nu_i', a\sum_{i=1}^{n}\lambda_i\rangle \in G$ and therefore $\langle \Sigma\mu_i, \Sigma\lambda_i\rangle \in \Phi$. Thus Φ satisfies the first part of condition (3b). It is easy to see that the second part is satisfied as well.

III. Let $\mu_i \in \mathcal{M}(P)$, $i = 0,\ldots,n$, $\mu_0 = \sum_{i=1}^{n}\mu_i$, $\langle \mu_0 , \lambda_0\rangle \in \Phi$. Then, for some $\nu_0 \in \mathcal{M}(S)$ and some $a>0$, we have $\langle a\mu_0 , \nu_0\rangle \in F$, $\langle \nu_0 , a\lambda_0\rangle \in G$. Since (3d) holds for F, and $a\mu_0 = \sum_{i=1}^{n} a\mu_i$, there exist $\nu_i \in \mathcal{M}(S)$ such that $\langle a\mu_i, \nu_i\rangle \in F$, $i = 1,\ldots,n$, $\sum_{i=1}^{n}\nu_i = \nu_0$. Since (3d) holds for G, there exist $\lambda_i' \in \mathcal{M}(U)$ such that $\langle \nu_i , \lambda_i'\rangle \in G$, $i = 1,\ldots,n$, $\sum_{i=1}^{n}\lambda_i' = a\lambda_0$. Now put $\lambda_i = (1/a)\cdot\lambda_i'$. Then $\sum_{i=1}^{n}\lambda_i = (1/a)\cdot a\lambda_0 = \lambda_0$, hence (due to $\lambda_0 \in \mathcal{M}(U)$) $\lambda_i \in \mathcal{M}(U)$, $i = 1,\ldots,n$. Since $\lambda_i' = a\lambda_i$, we have $\langle \nu_i , a\lambda_i\rangle \in G$ and therefore, due to $\langle a\mu_i , \nu_i\rangle \in F$, $\langle \mu_i, \lambda_i\rangle \in \Phi$, $i = 1,2,\ldots,n$. Thus Φ satisfies (3d).

IV. Let $\langle \mu_i, \lambda_i\rangle \in \Phi$, $i = 1,2$. Then there exist $\nu_i \in \mathcal{M}(S)$, $a_i>0$, $i = 1,2$, such that $\langle a_i\mu_i , \nu_i\rangle \in F$, $\langle \nu_i , a_i\lambda_i\rangle \in G$. Since (3f) holds for F and G, we have $\hat{r}(a_1\mu_1, a_2\mu_2) = \hat{r}(\nu_1, \nu_2) = \hat{r}(a_1\lambda_1, a_2\lambda_2)$. This implies $\hat{r}(\mu_1, \mu_2) = \hat{r}(\lambda_1, \lambda_2)$.

(5) <u>Notation.</u> If there exists a conservative measure-correspondence $F:P \longrightarrow S$, we put $P \sim S$.

(6) The relation $\sim$ is an equivalence relation on $\{WM\}$. - This follows at once from (4) and from the fact that if $F: P \longrightarrow S$ is a conservative measure-correspondence, then so is

$F^{-1}: S \to$.

(7) <u>Definition.</u> If $P = \langle Q, \wp, \mu \rangle$, $S = \langle T, \sigma, \nu \rangle$ are FWM-spaces and $f: Q \to T$ is a mapping such that (i) $\nu t = \mu(f^{-1}t)$ for every $t \in T$, (ii) $\sigma(fq, fq') = \wp(q, q')$ for all $q, q' \in Q$, then $\langle f, P, S \rangle$, also denoted by $f: P \to S$, is called a conservative mapping.

(8) <u>Proposition.</u> If there exists a conservative mapping $f: P \to S$, then $P \mathrel{\dot{\sim}} S$.

<u>Proof.</u> Let $S = \langle T, \sigma, \nu \rangle$. For any $\mu' \in \mathcal{M}(P)$ let $F\mu'$ be the measure on T defined as follows: $(F\mu')Y = \mu'(f^{-1}(Y))$. Put $F = \{\langle \mu', F\mu' \rangle : \mu' \in \mathcal{M}(P)\}$. It is easy to prove that $\langle F, P, S \rangle$ is a conservative measure-correspondence.

(9) <u>Proposition.</u> If $P = \langle Q, \wp, \mu \rangle$, $S = \langle T, \sigma, \nu \rangle$ are FWM-spaces and $P \mathrel{\dot{\sim}} S$, then, for some FWM-space U, there exist conservative mappings $f: U \to P$, $f: U \to S$.

<u>Proof.</u> Clearly, we may assume $wP > 0$, $wS > 0$. Since $P \mathrel{\dot{\sim}} S$, there exists a conservative measure-correspondence $F: P \to S$. For convenience, we shall write $\mu' \sim \nu'$ instead of $\langle \mu', \nu' \rangle \in F$. For any $q \in Q$, let μ_q denote the measure on Q defined as follows: $\mu_q(q) = \mu(q)$, $\mu_q(q') = 0$ if $q' \in Q$, $q' \neq q$. Since $\mu = \sum_{q \in Q} \mu_q$, $\mu \sim \nu$, there exist, by (3d), measures $\nu^{(q)} \in \mathcal{M}(S)$ such that $\mu_q \sim \nu^{(q)}$, $\sum_{q \in Q} \nu^{(q)} = \nu$.

For any $t \in T$, let ν_t denote the measure on T defined as follows: $\nu_t(t) = \nu(t)$, $\nu_t(t') = 0$ if $t' \in T$, $t' \neq t$. Clearly, for every $q \in Q$, there are $a_{qt} \geqq 0$ such that

(I) $\nu^{(q)} = \sum_{t \in T} a_{qt} \nu_t$.

Since $\mu_q \sim \nu^{(q)}$, there exist, by (3d), $\mu^*_{qt} \in \mathcal{M}(P)$ such that $\mu^*_{qt} \sim a_{qt} \nu_t$, $\sum_{t \in T} \mu^*_{qt} = \mu_q$; clearly, there exist non-negative numbers b_{qt} such that $\mu^*_{qt} = b_{qt}\,\mu_q$, hence

(II) $b_{qt}\,\mu_q \sim a_{qt}\,\nu_t$ for all $q \in Q$, $t \in T$,

(III) $\sum_{t \in T} b_{qt}\,\mu_q = \mu_q$ for all $q \in Q$.

By (II) and (3e) we get

(IV) $b_{qt}\,\mu(q) = a_{qt}\,\nu(t)$ for all $q \in Q$, $t \in T$.

For any $q \in Q$, $t \in T$, we put $\lambda\langle q,t\rangle = b_{qt}\,\mu(q)$. The space $U = \langle V, \rho^*, \lambda\rangle$ is defined as follows: $V = \{\langle q,t\rangle \in Q \times T: \lambda\langle q,t\rangle > 0\}$, $\rho^*(\langle q_1,t_1\rangle, \langle q_2,t_2\rangle) = \rho(q_1,q_2)$, $\lambda(Y) = \sum(\lambda\langle q,t\rangle: \langle q,t\rangle \in Y)$. For any $v = \langle q,t\rangle \in V$, we put $f(v) = q$, $g(v) = t$.

By (III), for any $q \in Q$, $\mu(q) = \sum_{t \in T} b_{qt}\,\mu(q) = \sum(\lambda\langle q,t\rangle: \langle q,t\rangle \in V) = \lambda(g^{-1}q)$. If $v = \langle q_i,t_i\rangle \in V$, then $\rho^*(v_1,v_2) = \rho(q_1,q_2) = \rho(fv_1,fv_2)$. Thus f is a conservative mapping.

Since $\nu = \sum_{q \in Q} \nu^{(q)}$, we have, for any $t \in T$, $\nu(t) = \sum_{q \in Q} \nu^{(q)}(t)$, hence, by (I), $\nu(t) = \sum_{q \in Q} \sum_{y \in T} a_{qy}\,\nu_y(t) = \sum_{q \in Q} a_{qt}\,\nu(t)$ and therefore, by (IV), $\nu(t) = \sum_{q \in Q} \lambda\langle q,t\rangle = \lambda(g^{-1}t)$.

By (3f) and (II), we have, for any $q,x \in Q$, $t,y \in T$,

$$\hat{r}(b_{qt}\,\mu_q, b_{xy}\,\mu_x) = \hat{r}(a_{qt}\,\nu_t, a_{xy}\,\nu_y),$$

hence

$$\lambda\langle q,t\rangle\lambda\langle x,y\rangle\,\rho(q,x) = \lambda\langle q,t\rangle\lambda\langle x,y\rangle\,\sigma(t,y).$$

For $\langle q,t\rangle \in V$, $\langle x,y\rangle \in V$, this implies $\rho(q,x) = \sigma(t,y)$, hence $\rho^*(\langle q,t\rangle,\langle x,y\rangle) = \sigma(t,y)$. Thus g is conservative.

(10) The assertion (ES 1.16) that the relation $\sim$ (defined in ES 1.15) coincides on $\{FWM\}$ with the equivalence relation, also denoted by $\sim$, introduced in QE 1.4 (QE stands

for M. Katětov, Quasi-entropy of finite weighted metric spaces, Comment. Math. Univ. Carolinae 17(1976), 797-806) is to be replaced by the following one.

Proposition. The relation $\dot{\sim}$ coincides on $\{FWM\}$ with the equivalence relation introduced (and denoted by $\sim$) in QE 1.4.

Proof: follows at once from (8) and (9).

(11) The assertion (stated without proof in ES 3.7) that C^* is not invariant with respect to conservative morphisms (introduced in ES 1.13) is to be replaced by the following one: the function C^* is not invariant with respect to conservative measure-correspondences.

(12) The definition of a subentropy (ES 2.1) is to be changed by substituting $P_1 \dot{\sim} P_2$ for $P_1 \sim P_2$.

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