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ON MULTIVALUED MAPPINGS IN PARANORMED SPACES
Olga HADŽIĆ

Abstract: In Theorem 1 a sufficient condition for multivalued mapping $F:K \rightarrow 2^K$ ($K \subseteq E$ and E is a paranormed space) is given such that F has the finite approximation property [3] and in Theorem 2 that F has the almost continuous selection property, where K satisfies Zima's condition [6]. Also some corollaries in the fixed point theory are given.

Key words: Multivalued mappings, paranormed space, fixed point.

Classification: 47H10

Let E be a linear space over the real or complex number field. The function $\| \cdot \|^*: E \rightarrow [0, \infty)$ will be called a paranorm iff:

1. $\| x \|^{*} = 0 \iff x=0$
2. $\| -x \|^{*} = \| x \|^{*}$, for every $x \in E$
3. $\| x + y \|^{*} \leq \| x \|^{*} + \| y \|^{*}$, for every $x, y \in E$
4. If $\| x_n - x_0 \|^{*} \rightarrow 0$, $\lambda_n \rightarrow \lambda_0$ then $\| \lambda_n x_n - \lambda_0 x_0 \|^{*} \rightarrow 0$

Then we say that $(E, \| \cdot \|^{*})$ is a paranormed space. E is also a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\epsilon\}_{\epsilon > 0}$ where $U_\epsilon = \{x \mid x \in E, \| x \|^{*} < \epsilon\}$.

In [6] the following theorem is proved:

Let K be a bounded, closed and convex subset of E and $T:K \rightarrow K$ be a completely continuous operator on K . If there exists a number $C > 0$ such that:

(1) $\|\lambda x\|^* \leq C\lambda \|x\|^*$, for every $0 \leq \lambda \leq 1$ and $x \in K-K$ then there exists an element $p \in K$ such that $Tp=p$.

Zima has given in [6] an example of the space E and of the set K such that the condition (1) is satisfied.

Definition. Let $(E, \|\cdot\|^*)$ be a paranormed space and K be a non-empty subset of E . If there exists $C > 0$ such that:

$\|\lambda x\|^* \leq C\lambda \|x\|^*$, for every $0 \leq \lambda \leq 1$ and $x \in K-K$ we say that K satisfies the Zima condition.

In the next text we shall use the following notation. By 2^K ($K \subset E$) we shall denote the family of all non-empty subsets of the set K and by $\mathcal{R}(K)$ the family of all non-empty convex and closed subsets of the set K .

Now, we shall prove a theorem on the finite approximation property.

Theorem 1. Let $(E, \|\cdot\|^*)$ be a paranormed space, K be a non-empty, closed and convex subset of E and $F:K \rightarrow \mathcal{R}(K)$ be a closed mapping such that $F(K)$ is relatively compact and satisfies the Zima condition. Then for every $\varepsilon > 0$ there exists a finite dimensional, closed mapping $F_\varepsilon :K \rightarrow \mathcal{R}(K)$ such that $F_\varepsilon(K)$ is relatively compact and:

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon, \quad \forall x \in K$$

where: $V_\varepsilon = \{x | x \in E, \|x\|^* \leq \varepsilon\}$.

Proof: Since the set $F(K)$ is relatively compact, there exists a finite set $\{x_1, x_2, \dots, x_n\} \subset F(K)$ such that:

$$(2) \quad F(K) \subseteq \bigcup_{i=1}^n \{x_i + U_{\frac{\varepsilon}{C}}\} \quad (U_{\frac{\varepsilon}{C}} = \{x | x \in E, \|x\|^* < \frac{\varepsilon}{C}\})$$

Let:

$$F_\varepsilon(x) = [F(x) + \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K)))] \cap \overline{\text{co}} M$$

for every $x \in K$, where $M = \{x_1, x_2, \dots, x_n\}$.

For every $x \in K$ we have that $F_\varepsilon(x) \neq \emptyset$. Indeed, if $u \in F(x)$ it follows that there exists $x_i \in M$ and $z \in U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K))$ so that $u-z=x_i$ from which we conclude that:

$$u-z \in [F(x) + \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K)))] \cap \overline{\text{co}} M$$

(this follows from (2)).

Further $\overline{\text{co}} F_\varepsilon(x) = F_\varepsilon(x)$, for every $x \in K$ since $\overline{\text{co}} F(x) = F(x)$ for every $x \in K$, and $F_\varepsilon(K) \subseteq \overline{\text{co}} M$ which implies that the mapping F_ε is finite dimensional. Let us prove that the mapping F_ε is closed. Suppose that $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq K$ is a convergent net such that $\lim_{\alpha \in \mathcal{A}} x_\alpha = x$, $y_\alpha \in F_\varepsilon(x_\alpha)$, for every $\alpha \in \mathcal{A}$ and $\lim_{\alpha \in \mathcal{A}} y_\alpha = y$. We shall prove that $y \in F_\varepsilon(x)$ which means that the mapping F is closed. Since $y_\alpha \in F_\varepsilon(x_\alpha)$, for every $\alpha \in \mathcal{A}$ it follows that there exists $z_\alpha \in F(x_\alpha)$, for every $\alpha \in \mathcal{A}$ and $u_\alpha \in \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K)))$, for every $\alpha \in \mathcal{A}$ such that:

$$y_\alpha = z_\alpha + u_\alpha \in \overline{\text{co}} M$$

Since the set $\overline{F(K)}$ is compact there exists a convergent subnet $\{z_{\alpha_j}\}$ of the net $\{z_\alpha\}$ and let $\lim_{j} z_{\alpha_j} = z$. Since $y_{\alpha_j} \rightarrow y$ it follows that $u_{\alpha_j} = y_{\alpha_j} - z_{\alpha_j} \rightarrow u = y - z$. Further, the mapping F is closed and since $\lim_{j} x_{\alpha_j} = x$, $\lim_{j} z_{\alpha_j} = z$ and $z_{\alpha_j} \in F(x_{\alpha_j})$ it follows that $z \in F(x)$. From the relation:

$$u_{\alpha_j} \in \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K))), \text{ for every } j$$

it follows that $u \in \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K)))$ and so:

$$y = u + z \in (F(x) + \overline{\text{co}}(U_{\frac{\varepsilon}{c}} \cap (F(K)-F(K)))) \cap \overline{\text{co}} M = F_\varepsilon(x).$$

Now, we shall prove that for every $x \in K$ we have:

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon$$

Since $F_\varepsilon(x) \subseteq F(x) + \overline{\text{co}}(U_{\frac{\varepsilon}{C}} \cap (F(K) - F(K)))$ it remains to prove that:

$$(3) \quad \overline{\text{co}}(U_{\frac{\varepsilon}{C}} \cap (F(K) - F(K))) \subset V_\varepsilon$$

Suppose that $u \in \text{co}(U_{\frac{\varepsilon}{C}} \cap (F(K) - F(K)))$. Then $u = \sum_{i=1}^n t_i z_i$, where $\sum_{i=1}^n t_i = 1$, $t_i \geq 0$ ($i=1, 2, \dots, n$) and $z_i \in U_{\frac{\varepsilon}{C}} \cap (F(K) - F(K))$.

So we have:

$$\|u\|^* = \left\| \sum_{i=1}^n t_i z_i \right\|^* \leq C \sum_{i=1}^n t_i \frac{\varepsilon}{C} = \varepsilon$$

and so $u \in V_\varepsilon$. Since V_ε is closed it follows that the relation (3) is proved. It is obvious that $F_\varepsilon(K)$ is relatively compact since $F_\varepsilon(K) \subseteq \overline{\text{co}}\{x_1, x_2, \dots, x_n\}$, and so the proof is complete.

From Theorem 1 it is easy to obtain the following Corollary.

Corollary 1. Let $(E, \|\cdot\|^*)$ be a paranormed space, K be a non-empty, closed and convex subset of E and $F:K \rightarrow \mathcal{R}(K)$ be a closed mapping such that $F(K)$ is relatively compact and satisfies the Zima condition. Then there exists $x \in K$ such that $x \in F(x)$.

Proof: From Theorem 1 it follows that there exists, for every $\varepsilon > 0$, a compact finite dimensional mapping $F_\varepsilon:K \rightarrow \mathcal{R}(K)$ such that:

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon, \quad \forall x \in K$$

and that $F_\varepsilon(M) \subseteq M$ where M is $\overline{\text{co}}\{x_1, x_2, \dots, x_n\}$ (see Theorem 1). If we apply Kakutani's fixed point theorem we conclude that for every $\varepsilon > 0$ there exists x_ε such that $x_\varepsilon \in F_\varepsilon(x_\varepsilon)$ and so:

$$(4) \quad x_\varepsilon \in F(x_\varepsilon) + V_\varepsilon$$

Since the set $\overline{F_\varepsilon(K)}$ is compact there exists a sequence $(\varepsilon_n \rightarrow 0)$ $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $x_{\varepsilon_n} \rightarrow x \in K$ and since F is closed it is easy to see that $x \in F(x)$.

Remark: From Theorem 1 it follows that every closed and convex subset of E which satisfies the Zima condition is \mathcal{G} -admissible [4] and so we can apply a result of S. Hahn in order to obtain a fixed point theorem for multivalued mapping [4].

Corollary 2. Let $(E, \|\cdot\|)$ be a paranormed space, W be a closed neighbourhood of $b \in E$, K be a closed, convex subset of E and satisfies the Zima condition. Let $F: W \cap K \rightarrow \mathcal{R}(K)$ be a compact mapping such that:

$$x \in \partial W \cap K, \beta > 1 \Rightarrow \beta x + (1 - \beta)b \notin F(x)$$

Then there exists a point $x_0 \in W \cap K$ such that $x_0 \in F(x_0)$.

Now, we shall prove a theorem on almost continuous selection property for multivalued mapping in paranormed space.

First, we shall give a definition [2], introducing the notion of uniformly u -continuous mapping.

Definition 2. Let X be a topological vector space, K be a non-empty subset of X , $F: K \rightarrow 2^X$ and \mathcal{U} be the fundamental system of symmetric neighbourhoods of zero in X . The mapping F is uniformly u -continuous iff for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that the following implication holds:

$$x_1, x_2 \in K, x_1 - x_2 \in U \text{ and } y_1 \in F(x_1) \Rightarrow \text{there exists } y_2 \in F(x_2), \\ y_1 - y_2 \in V$$

Definition 3 ([1]). Let X be a topological vector space, \mathcal{U} be the fundamental system of neighbourhoods of zero in X ,

K be a non-empty subset of X and $F:K \rightarrow \mathcal{R}(K)$. The mapping F has the almost continuous selection property iff for every $V \in \mathcal{U}$ there exists a continuous mapping $h_V:K \rightarrow K$ such that:

$$h_V(x) \in F(x) + V, \text{ for every } x \in K$$

Theorem 2. Let $(E, \| \cdot \|^*)$ be a paranormed space, K be a compact and convex subset of E which satisfies the Zima condition. Then every uniformly u-continuous mapping $F:K \rightarrow \mathcal{R}(K)$ has the almost continuous selection property.

Proof: Let $\varepsilon > 0$. Since the mapping F is uniformly u-continuous on K there exists $\sigma > 0$ such that the following implication holds:

$$x_1, x_2 \in K, x_1 - x_2 \in U_\sigma, y_1 \in F(x_1) \Rightarrow \text{there exists } y_2 \in F(x_2), \\ y_1 - y_2 \in V_\varepsilon$$

From the compactness of the set K it follows that there exists $\{x_1, x_2, \dots, x_n\} \subseteq K$ such that:

$$K \subseteq \bigcup_{i=1}^n \{x_i + U_\sigma\}$$

and let $\{g_i\}_{i=1}^n$ be the partition of the unity subordinated to the open cover $\{x_i + U_\sigma\}_{i=1}^n$. Then $g_i(x) = 0$ if $x \notin x_i + U_\sigma$ ($i = 1, 2, \dots, n$), $g_i(x) \geq 0$ ($x \in K, i = 1, 2, \dots, n$) and $\sum_{i=1}^n g_i(x) = 1$. Let us define the mapping $h_\varepsilon : K \rightarrow K$ in the following way:

$$h_\varepsilon(x) = \sum_{i=1}^n g_i(x) y_i, \text{ for every } x \in K$$

where $y_i \in F(x_i)$ ($i = 1, 2, \dots, n$). Let us prove that:

$$(5) \quad h_\varepsilon(x) \in F(x) + V_\varepsilon, \text{ for every } x \in K$$

i.e. that for every $x \in K$ there exists $z(x) \in F(x)$ such that $h_\varepsilon(x) - z(x) \in V_\varepsilon$. Let $x \in K$ and $g_i(x) > 0$. Then $x - x_i \in U_\sigma$ and so there exists $u_i \in F(x)$ such that $y_i - u_i \in V_\varepsilon$, since F is uniformly u-continuous and $y_i \in F(x_i)$. Let:

$$z(x) = \sum_{i: g_i(x) > 0} g_i(x) u_i$$

Then $z(x) \in F(x)$ since $F(x) \in \mathcal{R}(K)$. Further:

$$\begin{aligned} \|h_\varepsilon(x) - z(x)\|^* &= \left\| \sum_{i: g_i(x) > 0} g_i(x) (y_i - u_i) \right\|^* \leq \\ &\leq \sum_{i: g_i(x) > 0} g_i(x) \cdot \frac{\varepsilon}{C} = \varepsilon \end{aligned}$$

and so the proof of (5) is complete.

In the next Corollary we shall use the notion of ε -fixed point of the multivalued mapping G in the following sense.

Definition 4. Let $(E, \|\cdot\|^*)$ be a paranormed space, K be a non-empty subset of E and $F: K \rightarrow 2^E$. Then $x \in K$ is an ε -fixed point of the mapping F iff:

$$x_\varepsilon \in F(x_\varepsilon) + V_\varepsilon$$

Corollary 3. Let $(E, \|\cdot\|^*)$ be a paranormed space, K be a compact, convex subset which satisfies the Zima condition. Then for every $\varepsilon > 0$ and every uniformly u -continuous multivalued mapping $F: K \rightarrow \mathcal{R}(K)$ there exists ε -fixed point of the mapping F .

Proof: Since from Theorem 2 it follows that there exists $h_\varepsilon: K \rightarrow K$, with (4), such that h_ε is continuous and $h_\varepsilon: K \rightarrow \overline{CO} \{x_1, x_2, \dots, x_n\}$ from Brouwer fixed point theorem it follows that there exists $x_\varepsilon \in K$ such that $x_\varepsilon = h_\varepsilon(x_\varepsilon)$ and so:

$$x_\varepsilon = h_\varepsilon(x_\varepsilon) \in F(x_\varepsilon) + V_\varepsilon$$

which means that x_ε is an ε -fixed point of the mapping F .

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