Jaroslav Ježek; Tomáš Kepka

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FREE ENTROPIC GROUPOIDS

Jaroslav JEŽEK, Tomáš KEPKA

Abstract: Free entropic groupoids and free commutative medial groupoids are constructed.

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The theory of medial groupoids is (at least in the opinion of the authors) one of the deepest non-associative theories within the framework of groupoids. (Recall that a groupoid is said to be medial if it satisfies the identity $xy.uv = xu.yv$ and a groupoid is said to be entropic if it is a homomorphic image of a medial cancellation groupoid.) In this paper, explicit constructions of free entropic and free commutative medial groupoids are presented. These constructions are realized by means of polynomials in at most two commuting indeterminates over the ring of integers.

1. Preliminaries. Let us fix two symbols, say α and β. We denote by E the free monoid over $\{\alpha, \beta\}$. Every element $e \in E$ can be written in the form $e = a_1 \ldots a_n$, $a_i \in \{\alpha, \beta\}$

and $n \geq 0$; the integer n is denoted by $\delta(e)$. The unit element of E is 1 and we have $\delta(1) = 0$. For every $n \geq 0$, let $E_n = \{e \in E;\ \delta(e) = n\}$.

Let X be a non-empty set. Then SW'_X is the free algebra over X in the variety of universal algebras of the type $\{+, 0, \alpha, \beta\}$ (consisting of one binary, one nullary and two unary operation symbols) satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$. Elements from SW'_X are called semiterms over X. Every semiterm can be expressed in the form $s = \sum_{i=1}^{r} e_i x_i$, where r is a non-negative integer, $e_i \in E$ and $x_i \in X$; this expression is unique up to the order of the summands. We put $\lambda(s) = r$.

Define a multiplication on SW'_X by $st = \alpha s + \beta t$. The set SW'_X is a groupoid with respect to this operation and this groupoid will be denoted by SW_X. Let W_X be the subgroupoid of SW_X generated by X. Elements of W_X are called terms over X. It is easy to check that W_X is an absolutely free groupoid over X.

Let $t = \sum_{i=1}^{r} e_i x_i$ be a term over X. The set $\{x_i; 1 \leq i \leq r\}$ is denoted by var(t). The set $\{e_i; 1 \leq i \leq r\}$ is denoted by $I^*(t)$. The set $\{e \in E; ef \in I^*(t)$ for some $f \in E\}$ is denoted by I(t). For every $n \geq 0$, put $I_n(t) = E_n \cap I(t)$. Finally, let $\delta(t) = \max\{\delta(e); e \in I(t)\}$.

1.1. <u>Lemma</u>. Let t be a term over X. The set $P = I(t)$ has the following properties:

(1) P is a finite subset of E and $1 \in P$.

(2) If $e, f \in E$ and $ef \in P$ then $e \in P$.

(3) If $e \in E$ then $e\alpha \in P$ iff $e\beta \in P$.

Conversely, if P is a subset of E satisfying (1),(2) and (3) and h is a mapping of the set $Q = \{e \in P; e\alpha \notin P\}$ into X then the semiterm $s = \sum_{e \in Q} eh(e)$ is a term over X and $P = I(s)$.

Proof. Easy (by induction on $\lambda(t)$).

Let t be a term over X and $e \in I(t)$. Then, as one may check easily, there exists a unique pair (w,u) such that w is a semiterm, u is a term and $t = w + eu$. Moreover, if v is a term then $w + ev$ is a term. We put $u = t_{[e]}$.

Let $e = a_1 \ldots a_n \in E$. The ordered pair (Card $\{i; a_i = \alpha\}$, Card $\{i; a_i = \beta\}$) is called the weight of e. For every pair (k,l) of non-negative integers, denote by $E_{k,l}$ the set of all $e \in E$ of weight (k,l). For every $t \in W_X$, let $I_{k,l}(t) = E_{k,l} \cap \cap I(t)$. Finally, put $P_{k,l}(x,t) = \mathrm{Card}\,\{e \in I_{k,l}(t); t_{[e]} = x\}$ for each $x \in X$.

1.2. <u>Proposition</u>. Let X be a non-empty set. Denote by M_X the least congruence of W_X such that the corresponding factorgroupoid is a medial cancellation groupoid. Then $(u,v) \in \in M_X$ iff $u,v \in W_X$ and $P_{k,l}(x,u) = P_{k,l}(x,v)$ for all $x \in X$ and $k,l \geq 0$.

Proof. See [2].

For all $t \in W_X$, $x \in X$ and $n \geq 0$, put $P_n(x,t) = \mathrm{Card}\,\{e \in \in I_n(t); t_{[e]} = x\}$.

1.3. <u>Proposition</u>. Let X be a non-empty set. Denote by C_X the least congruence of W_X such that the corresponding factorgroupoid is a commutative medial groupoid. Then $(u,v) \in \in C_X$ iff $u,v \in W_X$ and $P_n(x,u) = P_n(x,v)$ for all $x \in X$ and $n \geq 0$.

Proof. See [2].

In the following, we shall make use of the numbers $\binom{n}{k}$. Recall that these are defined as follows: $\binom{n}{k} = 0$ for all $n < 0$ and every integer k; $\binom{0}{0} = 1$; $\binom{0}{k} = 0$ for every $k \neq 0$; $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for all $n \geq 0$ and every integer k.

2. <u>A construction of free entropic groupoids</u>. Let X be a non-empty set. Denote by F'_X the free algebra over X in the variety of universal algebras of the type $\{+, 0, \alpha, \beta\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\beta(x + y) = \beta x + \beta y$, $\alpha 0 = 0$, $\beta 0 = 0$, $\alpha\beta x = \beta\alpha x$. Every element $u \in F'_X$ can be written in the form $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ where r, n_i, m_i are non-negative integers and $x_i \in X$; this expression is unique up to the order of the summands. We define a multiplication on F'_X by $uv = \alpha u + \beta v$. The set F'_X together with this operation is a groupoid which will be denoted by F_X. We can identify the set F_X with a subset of SW_X. For every element $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ from F_X, put $\delta(u) = \max (n_i + m_i)$. This non-negative number is called the depth of u. Finally, denote by G_X the subgroupoid of F_X generated by X.

2.1. <u>Theorem</u>. Let X be a non-empty set. Then the groupoid G_X is a free entropic groupoid over X. An element $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ from F_X belongs to G_X iff the following two conditions are satisfied:

(1) If $0 \leq k \leq n \leq \delta(u)$ then $\sum_{i=1}^{r} \binom{n-n_i-m_i}{k-n_i} \leq \binom{n}{k}$.

(2) If $0 \leq k \leq n = \delta(u)$ then $\sum_{i=1}^{r} \binom{n-n_i-m_i}{k-n_i} = \binom{n}{k}$.

The proof of this result will be divided into seven lemmas.

2.2. <u>Lemma</u>. Denote by h the homomorphism of W_X onto G_X such that $h(x) = x$ for every $x \in X$. If $t \in W_X$ then $h(t) = \sum_{e \in I^*(t)} \alpha^{l(e)} \beta^{r(e)} t_{[e]}$ where $(l(e), r(e))$ denotes the weight of e. For $u, v \in W_X$, $h(u) = h(v)$ iff $P_{k,l}(x,u) = P_{k,l}(x,v)$ for all $x \in X$ and $k, l \geq 0$.

Proof. The first assertion can be proved easily by induction on the length of t. The second assertion is an easy consequence.

2.3. <u>Lemma</u>. G_X is a free entropic groupoid over X.

Proof. This is a consequence of 2.2 and 1.2.

Let $u = \sum_{i=1}^{k} \alpha^{n_i} \beta^{m_i} x_i \in F_X$. Put $c(n,k,u) = \sum_{i=1}^{k} \binom{n-n_i-m_i}{k-n_i}$ for all integers n, k. Let H_X designate the set of all $u \in F_X$ such that $c(n,k,u) = \binom{n}{k}$ whenever $0 \leq k \leq n = \sigma(u)$. Moreover, let K_X be the set of all $u \in H_X$ such that $c(n,k,u) \leq \binom{n}{k}$ whenever $0 \leq k \leq n \leq \sigma(u)$.

2.4. <u>Lemma</u>. The following conditions are equivalent for every $u \in F_X$:

(i) $u \in H_X$.

(ii) $c(n,k,u) = \binom{n}{k}$ for all $n \geq \sigma(u)$ and k.

(iii) There exists $n \geq \sigma(u)$ such that $c(n,k,u) = \binom{n}{k}$ for every $0 \leq k \leq \sigma(u)$.

Proof. (i) implies (ii). If either $k < 0$ or $n > k$ then $c(n,k,u) = 0 = \binom{n}{k}$. Now, we shall proceed by induction on $m = n - \sigma(u)$. If $m = 0$ then there is nothing to prove. Let $m \geq 1$. We have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and $\binom{n-1}{k-1} = c(n-1,k-1,u)$, $\binom{n-1}{k} = c(n-1,k,u)$ by the induction hypothesis. The rest is clear.

(ii) implies (iii). This implication is evident.

(iii) implies (i). We can assume that $n > \sigma(u)$. Let us prove by induction on $0 \le k \le \sigma(u)$ that $c(n-1,k,u) = \binom{n-1}{k}$. For $k = 0$, $1 = c(n,0,u) = \binom{n}{0}$, and hence $n_i = 0$ for exactly one $1 \le i \le r$; for this i we have $n - n_i - m_i \ge 1$, and so $c(n-1,0,u) = 1 = \binom{n-1}{0}$. Now, let $k \ge 1$. Then $\binom{m-1}{k} = \binom{n}{k} - \binom{m-1}{k-1} = c(n,k,u) - c(n-1,k-1,u) = c(n-1,k,u)$. The rest is easy by induction on $n - \sigma(u)$.

2.5. <u>Lemma</u>. $c(n,k,u) \le \binom{n}{k}$ for every $u \in K_X$ and all integers n, k.

Proof. The statement is clear, provided either $n < 0$ or $n \ge 0$ and $k \notin \{0,\dots,n\}$. Let $n \ge 0$ and $0 \le k \le n$. For $n \le \sigma(u)$, there is nothing to prove. For $n \ge \sigma(u)$, the assertion follows from 2.4.

2.6. <u>Lemma</u>. $G_X \subseteq K_X$.

Proof. Since $X \subseteq K_X$, it suffices to show that K_X is a subgroupoid of F_X. However, this is an easy consequence of 2.4 and 2.5.

2.7. <u>Lemma</u>. Let $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i \in K_X$ and $u \in X$. Then $n_i + m_i > 0$ for every $1 \le i \le r$.

Proof. Suppose, on the contrary, that $n_j = m_j = 0$ for some $1 \le j \le r$. Since $u \in X$, we have $r \ge 2$. Let $1 \le p \le r$. Then $\binom{n_p+m_p}{n_p} \ge c(n_p+m_p, n_p, u) \ge \binom{n_p+m_p-n_p-m_p}{n_p-n_p} + \binom{n_p+m_p-n_j-m_j}{n_p-n_j} = 1 + \binom{n_p+m_p}{n_p}$, since $u \in K_X$, a contradiction.

2.8. <u>Lemma</u>. $K_X \subseteq G_X$.

Proof. Consider an element $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ from K_X. For every $0 \le n \le \sigma(u)$ and every integer k, we shall construct by induction on n two sets A_k^n and B_k^n having the following

(1) $A_k^n \cap B_k^n = \emptyset$.

(2) Card $(A_k^n) = \binom{n}{k} - c(n,k,u)$.

(3) Card $(B_k^n) = \text{Card}\,\{i; n_i = k,\ m_i = n-k\}$.

(4) If $0 \leq k \leq n$ then A_k^n and B_k^n are subsets of $E_{k,n-k}$.

First, let $n = 0 = k$. If $u \in X$, then put $A_0^0 = \emptyset$ and $B = \{\emptyset\}$. If $u \notin X$, then put $A_0^0 = \{0\}$ and $B_0^0 = \emptyset$. Further, let either $k < 0$ or $k > n$. Then we put $A_k^n = B_k^n = \emptyset$. Finally, let $n > 0$ and $0 \leq k \leq n$. Denote by I the set of all $a_1 \ldots a_n \in E_{k,n-k}$ such that either $a_n = \alpha$ and $a_1 \ldots a_{n-1} \in A_{k-1}^{n-1}$ or $a_n = \beta$ and $a_1 \ldots a_{n-1} \in A_k^{n-1}$. By the induction hypothesis, Card (I) $=$ Card (A_{k-1}^{n-1}) + Card $(A_k^{n-1}) = \binom{n-1}{k-1} - c(n-1,k-1,u) + \binom{n-1}{k} - c(n-1,k,u) = \binom{n}{k} - c(n,k,u) + \sum_{i; n_i+m_i<n} \binom{n-n_i-m_i}{k-n_i} = \binom{n}{k} - c(n,k,u)$ + Card $\{i; n_i = k,\ m_i = n-k\}$. Since $\binom{n}{k} \geq c(n,k,u)$, the set I can be divided into two disjoint sets A and B of cardinalities $\binom{n}{k} - c(n,k,u)$ and Card $\{i; n_i = k,\ m_i = n-k\}$, resp. Now, it suffices to put $A_k^n = A$ and $B_k^n = B$.

Denote by C the union of all the sets A_k^n and B_k^n. Moreover, let D designate the union of all the sets B_k^n and let D_0 be the set of all $a_1 \ldots a_n \in C$ such that $a_1 \ldots a_n \alpha \notin C$. It is easy to see that $D \subseteq D_0$. As for the converse inclusion, let $a = a_1 \ldots a_n \in D_0$. We have $a \in E_{k,n-k}$ for some $0 \leq k \leq n$. If $n < \delta(u)$ then $a \in A_k^n$, and therefore $a \in B_k^n \subseteq D$. If $n = \delta(u)$ then $u \in K_X$ yields Card $(A_k^n) = \binom{n}{k} - c(n,k,u) = 0$, $A_k^n = \emptyset$ and $a \in B_k^n \subseteq D$. We have proved that $D = D_0$. Further, it is easy to verify that the set C satisfies the conditions (1),(2) and (3) of 1.1. Moreover, Card (D) $= r$ and there exists a bijective mapping g of D onto $\{1,\ldots,r\}$ such that if $e \in B_k^n$ and term t over X such that $C = I(t)$ and $t_{[e]} = x_{g(e)}$ for every $e \in D$. According to 2.2, $h(t) = u$ where h is the homomorphism of W_X onto G_X extending id_X, and so $u \in G_X$.

Theorem 2.1 follows now immediately from 2.3, 2.6 and 2.8.

2.9. <u>Lemma</u>. Let $u \in F_X$, $x,y,z \in X$ and $c,d \geq 0$. Then $u + \alpha^c \beta^d x \in H_X$ iff $u + \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z \in H_X$.

Proof. Let $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ and $n = \max(\sigma(u), c+d+1)$. By 2.4, $u + \alpha^c \beta^d x \in H_X$ iff $\binom{n}{k} = c(n,k,u) + \binom{n-c-d}{k-c}$ for all k. Similarly, $u + \alpha^{c+1} \beta^d y + \alpha^c \beta^{d+1} z$ belongs to H_X iff $\binom{n}{k} = c(n,k,u) + \binom{n-c-d-1}{k-c-1} + \binom{n-c-d-1}{k-c}$ for every k. The rest is clear.

2.10. <u>Lemma</u>. Let $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i$ H_X and $n = \sigma(u)$. Then $r \leq 2^n$. Moreover, $r = 2^n$ iff $n_i + m_i = n$ for every $1 \leq i \leq r$. In this case, $\binom{n}{k} = \mathrm{Card}\{i; n_i = k\}$ for every k.

Proof. Easy.

2.11. <u>Lemma</u>. Let $u = \sum_{i=1}^{r} \alpha^{n_i} \beta^{m_i} x_i \in H_X$ and $n = \sigma(u)$. Suppose that $r = 2^n$. Then $u \in G_X$.

Proof. By 2.10, $n = n_i + m_i$ for all i and $\binom{n}{k} = \mathrm{Card}\{i; n_i = k\}$ for every k. However, for each $0 \leq k \leq n$, we have $\binom{n}{k} = \mathrm{Card}\,(E_{k,n-k})$, and so there exists a bijective mapping f_k of $E_{k,n-k}$ onto $\{i; n_i = k\}$. Put $f = f_0 \cup \dots \cup f_n$, so that f is a bijective mapping of E_n onto $\{1,\dots,r\}$. Clearly, there exists unique $t \in W_X$ such that $I(t) = E_0 \cup \dots \cup E_n$ and $t_{[e]} = x_{f(e)}$ for every $e \in E_n$. By 2.2, $h(t) = u$, and hence $u \in G_X$.

2.12. <u>Remark</u>. As it is proved in [3], every medial cancellation groupoid can be imbedded into a medial quasigroup. Using this and some results from [1], it is possible to prove

2.3 directly and without an application of the equational theory of entropic groupoids.

3. <u>A construction of free commutative medial groupoids</u>. Let X be a non-empty set. Denote by CF'_X the free algebra over X in the variety of algebras of the type $\{+, 0, \alpha\}$ satisfying the identities $(x + y) + z = x + (y + z)$, $x + y = y + x$, $x + 0 = x$, $\alpha(x + y) = \alpha x + \alpha y$, $\alpha 0 = 0$. Further, define a multiplication on CF'_X by $uv = \alpha u + \alpha v$. We obtain thus a groupoid CF_X. Let CG_X be the subgroupoid of CF_X generated by X. Finally, for $u = \sum_{i=1}^{r} \alpha^{n_i} x_i \in CF_X$, define $\sigma(u) = \max(n_i; 1 \leq i \leq r)$.

3.1. <u>Theorem</u>. Let X be a non-empty set. Then the groupoid CG_X is the free commutative medial groupoid over X. An element $u = \sum_{i=1}^{r} \alpha^{n_i} x_i$ of CF_X belongs to CG_X iff $\sum_{k=0}^{\infty} 2^{-k} \mathrm{Card}\{i; n_i = k\} = 1$.

The proof of this theorem will be divided into seven lemmas.

3.2. <u>Lemma</u>. Let h be the unique homomorphism of W_X onto CG_X such that $h(x) = x$ for every $x \in X$. If $t \in W_X$ then $h(t) = \sum_{e \in I^*(t)} \alpha^{\delta(e)} t_{[e]}$. If $u, v \in W_X$ then $h(u) = h(v)$ iff $P_n(x,u) = P_n(x,v)$ for all $x \in X$ and $n \geq 0$.

Proof. Easy.

3.3. <u>Lemma</u>. CG_X is a free commutative medial groupoid over X.

Proof. Use 3.2 and 1.3.

3.4. <u>Lemma</u>. Denote by L_X the set of all $u = \sum_{i=1}^{r} \alpha^{n_i} x_i \in CF_X$ such that $\sum_{k=0} 2^{-k} \mathrm{Card}\{i; n_i = k\} = 1$. Let $v \in CF_X$,

$x, y, z \in X$ and $n \geq 0$. Then $v + \alpha^n x \in L_X$ iff $v + \alpha^{n+1} y +$ $+ \alpha^{n+1} z \in L_X$.

Proof. Easy.

3.5. <u>Lemma</u>. $CG_X \subseteq L_X$.

Proof. It is easy to check that L_X is a subgroupoid of CF_X.

3.6. <u>Lemma</u>. Let $u = \sum_{i=1}^{r} \alpha^{n_i} x_i \in L_X$ and $r = 1$. Then $u \in X$ and $n_i = 0$.

Proof. Obvious.

3.7. <u>Lemma</u>. Let $u = \sum_{i=1}^{r} \alpha^{n_i} x_i \in L_X$ and $r \geq 2$. Then $\mathrm{Card}\,\{i; n_i = \sigma(u)\} \geq 2$.

Proof. Easy.

3.8. <u>Lemma</u>. $L_X \subseteq CG_X$.

Proof. Consider an element $u = \sum_{i=1}^{r} \alpha^{n_i} x_i \in L_X$. We are going to show by induction on r that $u \in CG_X$. According to 3.6, we can assume that $r \geq 2$. Put $n = \sigma(u)$. By 3.7, there are two different numbers $1 \leq c$, $d \leq r$ with $n_c = n_d = n$. Let $v = \sum_{i \neq c,d} \alpha^{n_i} x_i + \alpha^{n-1} x_c$. It is easy to see that $v \in L_X$. Hence by the induction hypothesis $v \in CG_X$. Further, we have $v = h(t)$ for some $t \in W_X$. By 3.2, there is an $e \in I_{n-1}(t)$ with $t_{[e]} = x_c$. Now, it is easy to show that $h(w) = u$ for some $w \in W_X$.

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Matematicko-fyzikální fakulta
Universita Karlova
Sokolovská 83, 18600 Praha 8
Československo

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