Le Van Hot
On the differentiability of multivalued mappings. I.

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Abstract: The concept of H.T. Banks and M.Q. Jacobs [2] of differentials of multivalued mappings is extended from reflexive Banach spaces to locally convex spaces. Moreover, some properties of differentiable multivalued mappings are derived.

Key words: Locally convex spaces, differentiable mappings, multivalued mappings.

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1. Preliminaries. In this paper, we shall consider only real locally convex spaces. Let $X$ be a locally convex space (l.c.s.), whose topology $\mathcal{T}$ is induced by a family of continuous seminorm $P$. We denote the family of all bounded (bounded closed, bounded convex closed, respectively) non-empty subsets of $X$ by $\mathcal{B}(X)$ ($\mathcal{Q}(X)$, $\mathcal{Q}_0(X)$ resp.). For each $p \in P$ we define a pseudometric $d_p$ on $\mathcal{B}(X)$ by

$$d_p(A,B) = \inf \{\lambda > 0 : A \subseteq B + \lambda S_p \text{ and } B \subseteq A + \lambda S_p\}$$

$$= \max \{\sup_{x \in A} \inf_{y \in B} p(x-y), \sup_{y \in B} \inf_{x \in A} p(x-y)\}$$

where $S_p = \{x \in X | p(x) \leq 1\}$. We denote the closure of a set $A+B$ by $A^*B$. Put $\hat{\mathcal{Q}} = \mathcal{Q}_0(X) \times \mathcal{Q}_0(X) / \sim$, where the equivalence $\sim$ is defined by: $(A,B) \sim (C,D)$ iff $A^*D = B^*C$. Denote the class
containing \((A,B)\) by \([A,B]\) and define

\[
[A,B] + [C,D] = [A+C, A+D] \quad \text{for} \quad [A,B],[C,D] \in \hat{X},
\]

\[
\lambda [A,B] = [\lambda A, \lambda B] \quad \text{if} \quad \lambda \geq 0 \quad \text{and} \quad \lambda [A,B] = [-|A| B, |\lambda| A] \quad \text{if} \quad \lambda < 0.
\]

We use the following

**Embedding theorem [4].**

1) \( \hat{X} \) is a linear space.

2) The family \( \hat{P} = \{\hat{p} | p \in P\} \) of seminorms on \( \hat{X} \) defined by \( \hat{p}([A,B]) = dp (A,B) \) induces a locally convex topology \( \hat{\tau} \)
on \( \hat{X} \).

3) The map \( \varpi: \mathcal{C}_0(X) \to \hat{X} \) defined by \( \varpi(A) = [A,\{0\}] \) is isometric in the following sense: \( \hat{p}(\varpi(A) - \varpi(B)) = dp (A,B) \) for all \( A,B \in \mathcal{C}_0(X) \) and for continuous seminorms \( p \) on \( X \).

Now we turn to the definition of differentiability of multivalued mappings.

Let \( M \) be a set and let \( F \) be a map of \( M \) into \( \mathcal{C}_0(X) \); then we define a map \( \hat{F} \) of \( M \) into \( \hat{X} \) by:

\[
\hat{F}(m) = \varpi(F(m)) = [F(m),\{0\}] \quad \text{for all} \quad m \in M.
\]

If \( F \) is a map of \( M \) into \( \hat{X} \), then it is clear that there exist maps \( A, B \) of \( M \) into \( \mathcal{C}_0(X) \) such that \( F(m) = [A(m),B(m)] \) for all \( m \in M \) and we write \( F = [A,B] \).

**Definition 1.** Let \( X, Y \) be locally convex spaces. A map \( F \) of \( X \) into \( \mathcal{C}_0(Y) \) is said to be positively homogeneous if \( F(tx) = tf(x) \) for all \( x \in X \) and \( t \geq 0 \).

In the remainder of this section we always suppose that \( X, Y \) are locally convex spaces, \( \Omega \) is an open subset of \( X \), \( F \) is a map of \( \Omega \) into \( \mathcal{C}_0(Y) \).
Definition 2. (H.T. Banks and Q.M. Jacobs [2].) The mapping \( F \) is said to be directionally differentiable at \( x_0 \in \Omega \) iff the mapping \( \hat{F} \) has directional derivatives in every direction \( h \) of \( X \); i.e. for each \( h \in X \) there exists

\[
\lim_{t \to 0^+} \frac{\hat{F}(x_0 + th) - \hat{F}(x_0)}{t} = D_+\hat{F}(x_0)(h).
\]

This means that there exist positively homogeneous maps \( A(x_0)(\cdot), B(x_0)(\cdot) \) of \( X \) into \( \mathcal{C}_0(Y) \) such that for each continuous seminorm \( p \) on \( Y \), for each \( h \in X \) and \( t > 0 \) such that \( x_0 + th \in \Omega \), the function \( \omega_p(h,t) \) defined by

\[
\omega_p(h,t) = dp(F(x_0 + th) + B(x_0)(th), F(x_0) + A(x_0)(th))
\]

satisfies the condition

\[
\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0.
\]

If \( D_+\hat{F}(x_0) = [A(x_0), B(x_0)] \in L(X, \hat{Y}) \) and

\[
\lim_{t \to 0^+} \frac{\omega_p(h,t)}{t} = 0
\]

uniformly with respect to \( h \) on each bounded subset of \( X \) for each continuous seminorm \( p \) on \( Y \), then \( F \) is said to be Fréchet differentiable at \( x_0 \); in this case we write \( D\hat{F}(x_0)(h) = D_+\hat{F}(x_0)(h) \).

We say that \( F \) is strictly conically differentiable at \( x_0 \) if \( F \) is directionally differentiable at \( x_0 \) and \( D_+\hat{F}(x_0)(h) \in \mathcal{C}_0(Y) \) for each \( h \in X \); i.e. there exists a positively homogeneous map \( A(x_0) \) of \( X \) into \( \mathcal{C}_0(Y) \) such that \( D_+\hat{F}(x_0)(h) = [A(x_0)(h), \{0\}] \) for all \( h \in X \). In this case we write \( D\hat{F}(x_0)(h) = A(x_0)(h) \).

2. Some properties of differentiable mappings. Throughout this section \( X, Y, Z \) denote locally convex spaces, \( \Omega \) an open subset of \( X \), \( F \) a map of \( \Omega \) into \( \mathcal{C}_0(Y) \). Let \( T \in L(X,Y) \).
and define maps $T_c: C_0(X) \rightarrow C_0(Y)$ and $\hat{T} \in L(\hat{X}, \hat{Y})$ by $T_c(A) = T(A)$ and $\hat{T}([A, B]) = [T_c(A), T_c(B)]$ for $A \in C_0(X)$ and $[A, B] \in C_0(\hat{X})$ (see [5]).

**Lemma 1.** Let $T \in L(Y, Z)$ be given and let $F$ be directionally differentiable at $x_0$. Then the map $T_c \circ F$ of $\Omega$ into $C_0(Z)$, defined by $(T_c \circ F)(x) = T_c(F(x))$ for all $x \in \Omega$, is directionally differentiable at $x_0$ and $D^+_c(T_c \circ F)(x_0)(h) = \hat{T}(D^+_cF(x_0)(h))$.

If $F$ is strictly conically differentiable at $x_0$, then $T_c \circ F$ is also strictly conically differentiable at $x_0$ and

$$D^+_c(T_c \circ F)(x_0)(h) = T_c(D^+_cF(x_0)(h)).$$

**Proof.** The proof is obvious, since $T_c \circ F = \hat{T} \circ \hat{F}$.

**Theorem 1.** Suppose that $F$ is directionally differentiable at $x_0$ and $D^+_cF(x_0)(h) = [A(x_0)(h), B(x_0)(h)]$. Assume that $F$ satisfies the following condition:

(1) There exists a map $C$ of $\Omega$ into $C_0(Y)$ such that for each continuous seminorm $p$ on $Y$ and for each $h \in X$, and each $t > 0$ such that $x_0 + th \in \Omega$ we have $\lim_{t \to 0^+} \frac{\omega_p(h, t)}{t} = 0$, where

$$\omega_p(h, t) = dp(F(x_0 + th), F(x_0) + C(x_0 + th)).$$

Then $F$ is strictly conically differentiable at $x_0$ if one of the following two conditions is satisfied:

a) $Y$ is a semireflexive space or a space of the type $LL$,
b) for each $h \in X$, one of the sets $A(x_0)(h), B(x_0)(h)$ is weakly compact.

Moreover, if $Y$ is normable and each map $F$ which is directionally differentiable at $x_0$ and satisfies the condition (1), is strictly conically differentiable at $x_0$, then $Y$ is - 270 -
Proof. 1. The condition (1) can be written as follows:

\[ p(F(x_0 + th) - F(x_0) - C(x_0 + th)) \leq \omega_p(h, t) = O(t) \text{ if } t \to 0^+. \]

Then

\[ D_+ \hat{F}(x_0)(h) = \lim_{t \to 0^+} \frac{\hat{F}(x_0 + th) - \hat{F}(x_0)}{t} = \lim_{t \to 0^+} \frac{\hat{C}(x_0 + th)}{t} \]

\[ = \lim_{n \to \infty} \frac{\hat{C}(x_0 + n^{-1}h)}{n^{-1}} \in \mathcal{L}_0(Y)^S \]

where \( \mathcal{L}_0^S \) denotes the sequential closure of the set \( \mathcal{L}_0 \). Now the assertion of the first part of our Theorem follows from Corollaries 1,4 [5].

2. Let \( Y \) be a normed space and let the assumption of the part 2 of Theorem be satisfied. We shall prove that the space \( Y \) coincides with the completion \( \overline{Y} \) of \( Y \). Let \( y \in \overline{Y} \) be given; then there exist \( y_n \in Y \) (\( n \geq 1 \)) such that \( y = \sum_1^\infty y_n \) and \( \sum_1^\infty \| y_n \| < \infty \). Put

\[
\alpha(t) = \begin{cases} 
1 - 3n|t| & \text{for } t: |t| \leq \frac{1}{3n} \\
\frac{1}{2} - \frac{3}{2} n|t| & \text{for } t: \frac{1}{3n} \leq |t| \leq \frac{2}{3n} \\
-\frac{3}{2} + \frac{3}{2} n|t| & \text{for } t: \frac{2}{3n} \leq |t| \leq \frac{1}{n} \\
0 & \text{for } t: |t| \geq \frac{1}{n} 
\end{cases}
\]

\[
\beta_n(t) = \int_0^t \alpha_n(\tau) d\tau \quad \text{for } n = 1,2,\ldots, \quad f(t) = \sum_1^\infty \beta_n(t)y_n \in \overline{Y}.
\]

Then it is easy to verify that \( f'(t) = Df(t)(1) = \sum_n \alpha_n(1)y_n \). Let \( h_0 \in X, \ h_0 \neq 0, \ X_1 = \{ t h_0 \mid t \in \mathbb{R} \} \). We define a map \( u \) of \( X_1 \) into \( \mathcal{L}_0(Y) \) by

\[ u(th_0) = \{ \sum_n \beta_n(t)y_n \} \in \mathcal{L}_0(Y) \ (as \ \beta_n(0) = 0 \text{ for all } n \text{ and for } t \neq 0, \ \beta_n(t) \neq 0 \text{ only for a finite number of } n). \]

Let \( i \) be
the inclusion of \( Y \) into \( \hat{Y} \). Then the map \( \hat{i} \circ u \) is Fréchet differentiable on \( X \), since 
\[
\hat{i} \circ u(th_0) = \{ f(t) \}, \{ \partial f \} \text{ and } D(\hat{i} \circ u)(th_0)(h_0) = \{ f'(t) \}, \{ \partial f' \}
\]
for all \( t \in \mathbb{R} \). Furthermore, we know that the map \( \hat{\iota} \) is an isomorphism of \( \hat{Y} \) onto \( \hat{Y} \) (see Remark 3 after Theorem 3 [5]). Hence the map \( \hat{u} = (\hat{\iota})^{-1}(\hat{i} \circ u) \) is Fréchet differentiable on \( \hat{Y} \). By the Definition 2, it follows that \( u \) is Fréchet differentiable on \( X \). Let \( \pi \) be the projection of \( X \) onto \( X \). We define a map \( F \) of \( \Omega \) into \( \mathscr{C}_0(Y) \) by 
\[
F(x) = u(\pi(x-x_0)) \text{ for all } x \in \Omega.
\]
Then, of course, \( F \) satisfies the condition (1) with \( C(x) = F(x) \), and \( F \) is Fréchet differentiable on \( \Omega \) (so at \( x_0 \)) and 
\[
D\hat{F}(x_0)(h) = D\hat{u}(\circ)(\pi h). 
\]
By the assumption, \( F \) is strictly conically differentiable at \( x_0 \), so there exists an \( A \in \mathscr{C}_0(Y) \) such that 
\[
D\hat{F}(x_0)(h_0) = \{ A, \{ 0 \} \}. 
\]
Then 
\[
\hat{\mathbb{A}} = \hat{1}(D\hat{F}(x_0)(h_0)) = \hat{1}(D\hat{u}(\circ)(h_0)) = \hat{1}(\hat{i} \circ u)(\circ)(h_0) = \{ y \}, \{ 0 \},
\]
where \( \hat{\mathbb{A}} \) denotes the closure of \( A \) in \( \hat{Y} \). Hence: \( y \in \{ y \} = \hat{\mathbb{A}} = \mathbb{A} \subseteq Y \). This means that \( \hat{Y} \subseteq Y \) and this completes the proof.

**Theorem 2.** (The mean value theorem.) Suppose that \( F \) is directionally differentiable on \( \Omega \), 
\[
D_+\hat{F}(x)(h) = \{ A(x)(h), B(x)(h) \}
\]
for \( x \in \Omega \), \( h \in \mathbb{X} \) and let \( x_0, x_1 \in \Omega \) be given such that 
\[
\{ tx_0 + (1-t)x_1 \mid 0 \leq t \leq 1 \} \subseteq \Omega .
\]
Put \( k = x_1 - x_0 \). Then:

1) If \( D_+\hat{F}(x_0 + tk)(k) = \{ A(x_0 + tk)(k), \{ 0 \} \} \in \mathscr{C}(\mathscr{C}_0(Y)) \) for all \( t \in [0,1] \) and if \( Y \) is a space of the type \( LF \), then there exists a set \( Q(x_0, x_1) \in \mathscr{C}_0(Y) \) such that 
\[
F(x_1) = F(x_0) + {\ast} \ast Q(x_0, x_1).
\]

2) If \( Y \) is a regular inductive limit of a sequence of metrizable locally convex spaces and \( M = \text{conv} \{ A(x_0 + tk)(k) \mid 0 \leq t \leq 1 \} \in \mathbb{R} \), then there exists a set \( Q(x_0, x_1) \in \mathscr{C}_0(Y) \) such that 
\[
F(x_1) = F(x_0) + {\ast} \ast Q(x_0, x_1).
\]
and N = \text{conv} \{B(x_0 + tk)(k) \mid 0 \leq t \leq 1\} are separable and weakly compact, then there exist sets \(A(x_0, x_1), B(x_0, x_1) \in \mathcal{C}_0(Y), A(x_0, x_1) \subseteq M, B(x_0, x_1) \subseteq N\) such that \(F(x_1) + * B(x_0, x_1) = F(x_0) + * A(x_0, x_1)\).

**Proof.** By the mean value theorem for singlevalued mappings (see [1]) it follows that:

\[ [F(x_1), F(x_0)] = \hat{F}(x_1) - \hat{F}(x_0) = \hat{F}(x_0 + k) - \hat{F}(x_0) \in \text{conv} \{ D_{+} \hat{F}(x_0 + tk)(k) \mid 0 \leq t \leq 1 \} . \]

1) Let \( Y = \lim_{n \to \infty} Y_n \) be a space of the type \( LF \) and let \( D_{+} \hat{F}(x_0 + tk(0)) = [A(x_0 + tk)(k), \{0\}] \in \mathcal{C}_0(\mathcal{C}_0(Y)) \) for all \( t \in [0,1] \). If we put \( G(t) = F(x_0 + tk) \) for \( t \in [0,1 + 2\theta] \), where \( \theta \) is a positive number such that \( x_0 + tk \in \Omega \) for all \( t \in [0,1 + 2\theta] \), then we obtain a map \( G \) of \([0,1 + 2\theta]\) into \( \mathcal{C}_0(Y) \) which is directionally differentiable on \([0,1 + 2\theta]\). It implies that \( \hat{G} \), so as \( G \), is continuous on \([0,1 + \theta]\). It is easy to verify that the set \( \bigcup tG(t) \bigcap [0,1 + \theta] = \bigcup tF(x_0 + tk) \bigcap [0,1 + \theta] \) is bounded in \( Y \). By Theorem 6.5 ([1]), chapt. II) there exists \( n_0 \) such that \( \bigcup tF(x_0 + tk) \bigcap [0,1 + \theta] \subseteq Y_n, \) i.e. \( \hat{F}(x_0 + tk) \subseteq \hat{\mathcal{C}}_{n_0}(Y_n) \) where \( \hat{\mathcal{C}}_{n_0}(Y_n) \) is the inclusion of \( \hat{Y}_{n_0} \) into \( \hat{Y} \), for all \( t \in [0,1 + \theta] \). Then, of course, we have \( [A(x_0 + tk)(k), \{0\}] = D_{+} F(x_0 + tk)(k) \subseteq \hat{\mathcal{C}}_{n_0}(Y_n) \) for all \( t \in [0,1] \). We claim that \( A(x_0 + tk)(k) \subseteq \mathcal{C}_0(Y_{n_0}) \). Suppose that it is not true, then there exist \( t_0 \in [0,1] \) and a point \( y \in A(x_0 + t_0 k)(k) \) such that \( y \notin Y_{n_0} \). Since \( Y_{n_0} \) is a closed subspace of \( Y \), there exists a convex circled closed \( 0 \)-neighborhood \( N \) in \( Y \) such that \( (y + 2N) \cap Y_{n_0} = \emptyset \) i.e. \( A(x_0 + t_0 k)(k) \notin Y_{n_0} + 2N \). Then it follows that \( ([A(x_0 + t_0 k)(k), \{0\}] + \hat{\mathcal{C}}_{n_0}(Y_{n_0})) = \emptyset \), where
\[ \mathcal{U} = \{ [A, B] \in \hat{Y} \mid A \subseteq B + N \text{ and } B \subseteq A + N \} \] is an \( O \)-neighborhood in \( \hat{Y} \) (see [5]). This contradicts the fact that \([A(x_0 + t_0 k)(k), \{0\}] \in \hat{\mathcal{I}}(Y_{n_0})\). This means that \( D_{x} \hat{F}(x_0 + tk)(k) = [A(x_0 + tk)(k), \{0\}] \in \hat{\mathcal{I}}_{n_0}(\mathcal{C}(Y_{n_0})) \) for all \( t \in [0,1] \). By the mean value theorem for singlevalued mappings we have that \([F(x_1), F(x_0)] \in \hat{\mathcal{I}}_{n_0}(\mathcal{C}(Y_{n_0}))\). But we know that \( \mathcal{C}(Y_{n_0}) \) is a complete subset of \( Y_{n_0} \) (see [4]), as \( Y_{n_0} \) is an \( F \)-space, and \( \hat{\mathcal{I}}_{n_0} \) is an isomorphism of \( Y_{n_0} \) into \( \hat{Y} \) (see [5]). Hence \([F(x_1), F(x_0)] \in \hat{\mathcal{I}}_{n_0}(\mathcal{C}(Y_{n_0}))\). Thus, there exists a set \( Q(x_0, x_1) \in \mathcal{C}(Y_{n_0}) \subseteq \mathcal{C}(Y) \) such that \([F(x_1), F(x_0)] = [Q(x_0, x_1), \{0\}]\).

Then

\[ F(x_1) = F(x_0) +* Q(x_0, x_1). \]

2. Put \( \mathcal{M} = \{ [A, B] \in \hat{Y} \mid A \subseteq M, B \subseteq N \} \), then \( \mathcal{M} \) is a convex subset of \( \hat{Y} \) and by Proposition 2 [5] is \( \lambda(Y, Y') \)-compact. Therefore \( \mathcal{M} \) is \( \lambda(Y, Y') \)-closed and it implies that \( \mathcal{M} \) is closed in \( \hat{Y} \) in topology \( \tau \), where \( \tau \) is the topology of \( Y \). It is clear now that \([F(x_1), F(x_0)] \in \mathcal{M} \), since \( D_{x} \hat{F}(x_0 + tk)(k) \in \mathcal{M} \) for all \( t \in [0,1] \). Therefore there exist sets \( A(x_0, x_1) \subseteq M, B(x_0, x_1) \subseteq N \) such that \([F(x_1), F(x_0)] = [A(x_0, x_1), B(x_0, x_1)]\), which means that \( F(x_1) +* B(x_0, x_1) = F(x_0) +* A(x_0, x_1) \).

This completes the proof.

Theorem 3. Suppose that \( F \) is strictly conically differentiable on \( \Omega \) (i.e. \( F \) is strictly conically differentiable at each point \( x \in \Omega \)). Then

1. \( D_{x} F(x)(h) \) is a singleton for all \( x \in \Omega \) and \( h \in X \);
2. if \( \Omega \) is connected and \( Y \) is quasi-complete, then for each \( x_0 \in \Omega \) there exists a unique singlevalued mapping \( f \) of
Ω into Y such that:

\[ F(x) = F(x_0) + f(x) \]

and

\[ D_*F(x)(h) = \left\{ D_*f(x)(h) \right\} \]

for all \( x \in \Omega \) ; \( h \in X \).

**Proof.** We divide the proof in two steps.

**Step I.** First of all we suppose that Y is the space of the type LF. For each \( x \in \Omega \) , we take a convex neighborhood \( U(x) \) of \( x \) contained in \( \Omega \) . By the mean value Theorem 2 for each \( z \in U(x) \) there exist sets \( A(x,z) \) and \( A(z,x) \) of \( C(Y) \) such that

\[ F(z) = F(x) + A(x,z) \text{ and } F(x) = F(z) + A(z,x). \]

Then

\[ F(z) = F(x) + A(x,z) + A(z,x) \text{ or } A(x,z) + A(z,x) = \{0\}. \]

The latter identity holds if and only if \( A(x,z), A(z,x) \) are singletons and \( A(x,z) = -A(z,x) \). Let \( g(x,z) \) be a unique element of \( A(x,z) \). Then \( F(z) = F(x) + g(x,z) \) for all \( z \in U(x) \). It is easy to verify that the map \( g(x_1, \cdot) \) of \( U(x) \) into Y is directionally differentiable on \( U(x) \) and \( D_*F(x)(h) = \{D_*g(x,x)(h)\} \) for all \( h \in X \). This shows that \( D_*F(x)(h) \) is a singleton for all \( x \in \Omega \) and \( h \in X \).

If \( \Omega \) is connected, put \( G = \{x; x \in \Omega \} \) and there exists a point \( f(x) \in Y \) such that \( F(x) = F(x_0) + f(x) \). One can verify that \( G \) is open and closed in \( \Omega \) . From connectedness of \( \Omega \) it follows that \( \Omega = G \) and it is clear that \( f \) is unique.

**Step II.** We denote the bidual space of Y by \( Y'' \) and let \( Y'' \) be endowed with the \( \mathcal{C} \)-topology \( \mathcal{C}'' \), where \( \mathcal{C} \) is the family of all equicontinuous subsets of \( Y' \). Then the canonical embedding (evaluation map) \( J \) of Y into \( Y'' \) is an isomorphism of Y into \( Y'' \). Let \( y' \in Y' \), then by Lemma 1 the mapping \( y_c \circ F \) is strictly conically differentiable on \( \Omega \) and by step I,
\( D^*_c(y^c \circ F) (x)(h) = y^c (D^*_c F(x)(h)) \) is a singleton for all \( x \in \Omega \), \( h \in X \), \( y \in Y \), since \( R \) is an \( F \)-space. It follows that \( D^*_c F(x)(h) \) is a singleton, since \( Y \) distinguishes points of \( X \). Denote the unique element of \( D^*_c F(x)(h) \) by \( u(x)(h) \), then for each \( x \in \Omega \), \( u(x)(\cdot) \) is a positively homogeneous mapping of \( X \) into \( Y \). If \( \Omega \) is connected, then for each \( y \in Y \) there exists a map \( f_y \) of \( \Omega \) into \( R \) such that \( y^c(F(x)) = y^c(F(x_0)) + f_y(x) \) and \( D^*_c f_y(x)(h) = y^c(u(x)(h)) \). Now we define a mapping \( v(x) \) of \( Y \) into \( R \) by \( v(x)(y) = f_y(x) \) for each \( x \in \Omega \). Then we claim that \( v(x) \in Y^* \) and \( D^*_c v(x)(h) = J(u(x)(h)) \) for each \( x \in \Omega \), \( h \in X \). We prove that

i) \( v(x) \) is a linear functional of \( Y \) into \( R \). Let \( y', z' \in Y \) and \( \alpha, \beta \in R \) be given, then

\[
D^*_c (\alpha y' + \beta z') = \alpha f_y - \beta f_z (x)(h) = D^*_c f_y'(x)(h) - \alpha D^*_c f_y(x)(h) - \beta D^*_c f_z(x)(h) = (\alpha y' + \beta z')(u(x)(h)) - \alpha y'(u(x)(h)) - \beta z'(u(x)(h)) = 0
\]

and \( f_{\alpha y' + \beta z'} - f_y' - \beta f_z(x_0) = 0 \).

It follows that \( f_{\alpha y' + \beta z'}(x) = \alpha f_y'(x) + \beta f_z(x) \) for all \( x \in \Omega \). Then \( v(x)(\alpha y' + \beta z') = v(x)(y') + \beta v(x)(z') \), i.e. \( v(x)(\cdot) \) is linear.

ii) \( v(x) \in Y^* \). For this purpose set

\[
V = (F(x) \cup F(x_0))^0 = \{ y' \in Y \mid \langle y', y \rangle \leq 1 \text{ for all } y \in F(x) \cup U \}
\]

Then \( V \) is an \( \varepsilon \)-neighborhood in the strong topology \( \beta(Y', Y) \) on \( Y \). For each \( y' \in V \) we have:

\[
| v(x)(y') | = | f_y(x) | = d(\{ f_y(x) \}, \{ 0 \}) = d(y^c(F(x_0)) + f_y(x), y^c(F(x_0))) = d(y^c(F(x_0)), y^c(F(x_0))) \leq 2.
\]

This shows that \( v(x) \) is a linear continuous functional on
iii) \( D^v(x)(h) = J(u(x)(h)) \) for all \( h \in X \). Let \( p'' \) be a continuous seminorm on \((Y'', \varepsilon''') \), \( S_{p''} = \{ y'' \in Y'': p''(y'') \leq 1 \} \). Then there exists an equicontinuous subset \( E \) of \( Y' \) such that \( S_{p''} = E^0 = \{ y'' \in Y'': |\langle y'', y' \rangle | \leq 1 \) for all \( y' \in E \). Let \( S_p = \{ y \in Y: |\langle y', y \rangle | \leq 1 \) for all \( y' \in E \) and let \( p \) be the gauge functional of the set \( S_p \). Then for each \( x \in \Omega \), \( h \in X \), \( t > 0 \), \( x + th \in \Omega \) and for each \( y' \in E \) we have:

\[
|\langle v(x + th) - v(x) - J(u(x)(th)), y' \rangle | = \omega_p(x + th) - \omega_p(x) - \omega_p(u(x)(th)) \leq \sup \{ |\langle v(x + th) - v(x) - J(u(x)(th)), y' \rangle |, y' \in E \} \leq \omega_p(h,t).
\]

And \( \lim_{t \to 0^+} \omega_p(h,t) = 0 \), since \( p \) is a continuous seminorm on \( Y \) and \( F \) is directionally differentiable at \( x \) and \( D^F(x)(h) = \omega_d(x)(h) \). Then \( p''(v(x + th) - v(x) - J(u(x)(th))) \leq \sup \{ |\langle v(x + th) - v(x) - J(u(x)(th)), y' \rangle |, y' \in E \} \leq \omega_p(h,t) \).

This means that \( D^v(x)(h) = J(u(x)(h)) \). Put \( G(x) = J_c(F(x)) - v(x) \). Then

\[
\hat{G}(x) = J_c - \hat{v}(x), \text{ where } \hat{v}(x) = \{ \{ v(x) \} \}, \quad D^\hat{G}(x)(h) = D^\hat{J}(J_c)(F(x)(h)) - (J(u(x)(h))) = 0.
\]

This means that \( \hat{G} \) (and simultaneously \( G \), is constant on \( \Omega \). This implies that \( J_c(F(x)) - v(x) = J_c(F(x_o)), \quad v(x) \in J_c(F(x)) - J_c(F(x_o)) \).

On the other hand, \( J_c(F(x)) = J(F(x)) = J(F(x)) = J(F(x)) \), since \( F(x) \) is a complete subset of \( Y \) and \( J \) is an isomorphism of \( Y \) into \( Y'' \). Then \( v(x) \in J(F(x)) - F(x_o) \) \( \subseteq J(Y) \). Put \( f(x) = \).
J^{-1}(v(x)), then \( F(x) = F(x_0) + f(x) \). Of course, \( D^\neq F(x)(h) = \{u(x)(h)\} = \{D^\neq f(x)(h)\} \). This completes the proof.

**Remark 1.** We can define the differentiation of the mapping \( F: \Omega \to \mathcal{B}(Y) \) in the same way as De Blasi [3].

**Definition 3.** The map \( F \) is said to be directionally differentiable at \( x_0 \in \Omega \) iff there exists a positively homogeneous map \( D_+F(x_0) \) of \( X \) into \( \mathcal{C}_0(X) \) such that for each continuous seminorm \( p \) on \( Y \) and for each \( h \in X \) and \( t > 0 \) such that \( x_0 + th \in \Omega \), we have

\[
\lim_{t \to 0^+} \frac{\omega p(h,t)}{t} = 0,
\]

where

\[
\omega p(h,t) = dp(F(x_0 + th), F(x_0) + D_+F(x_0)(th)).
\]

It is easy to see that if \( F \) is directionally differentiable at \( x_0 \), then the map \( \co F \) of \( \Omega \) into \( \mathcal{C}_0(Y) \) defined by \( \co F(x) = \overline{\conv F(x)} \) is strictly conically differentiable at \( x_0 \), and \( D_+(\co F)(x)(h) = D_+F(x)(h) \).

**Theorem 3'.** Let \( F: \Omega \to \mathcal{B}(Y) \) be directionally differentiable on \( \Omega \). Then: 1) \( D_+F(x)(h) \) is a singleton for all \( x \in \Omega \) and \( h \in X \); 2) if \( \Omega \) is connected and \( Y \) is quasicomplete, \( x_0 \in \Omega \), then there exists a unique map \( f \) of \( \Omega \) into \( Y \) such that

\[
F(x) = F(x_0) + f(x)
\]

\[
D_+F(x)(h) = \{D_+f(x)(h)\}
\]

for all \( x \in \Omega \) and \( h \in X \).

**Proof.** 1. By Theorem 3, \( D_+F(x)(h) = D_+(\co F)(x)(h) \) is a singleton for all \( x \in \Omega \), \( h \in X \).

2. By Theorem 3 there exists a map \( f \) of \( \Omega \) into \( Y \) such that \( (\co F)(x) = (\co F)(x_0) + f(x) \); \( D_+F(x)(h) = \{D_+f(x)(h)\} \). Set \( G(x) = F(x) - f(x) \). Let \( p \) be a continuous seminorm on \( Y \).
Put \( g(x) = \text{dp}(G(x), G(x_0)) \). Using the same arguments as in the proof of Theorem 3.2 [3], one can prove that \( D_x g(x)(h) = 0 \) for all \( x \in \Omega \), \( h \in X \). It follows then \( \text{dp}(G(x), G(x_0)) = 0 \) for all \( x \in \Omega \) and for all continuous seminorms \( p \) on \( Y \). This means that \( G(x) = G(x_0) = \overline{F(x_0)} \) and hence \( \overline{F(x)} = \overline{F(x_0)} + f(x) \).

**Remark 2.** If \( Y \) is not quasicomplete, then the second part of Theorem 3 is not true. For instance, we take a normed space which is not complete. Let \( \widetilde{Y} \) be the completion of \( Y \), \( y \in \widetilde{Y}, y \notin Y \). For each \( n \), choose \( z_n \in Y \) such that \( \|y - z_n\| \leq (4n^2)^{-1}2^{-n} \). Put \( y_1 = z_1, y_n = z_n - z_{n-1} \) for \( n = 1, 2, \ldots \).

Then

\[
\sum y_n = y, \quad \sum 4n^2 \|y_n\| < + \infty.
\]

Set

\[
\alpha_n(t) = \begin{cases} 
-1 + \frac{1}{n} + t & \text{for } 1 - \frac{1}{n} \leq t \leq 1 - \frac{1}{2n} \\
1 - t & \text{for } t: 1 - \frac{1}{2n} \leq t \leq 1 \\
0 & \text{for } t \leq 1 - \frac{1}{n} \text{ or } t \geq 1
\end{cases}
\]

Then \( \beta_n(t) = 0 \) for \( t \leq 1 - \frac{1}{n} \); \( \beta_n(t) = \frac{1}{4n^2} \) for \( t \geq 1 \).

Define

\[
f(t) = \sum 4n^2 \beta_n(t)y_n \\
F(t) = (f(t) + S_1) \cap Y \in C_0(Y),
\]

where \( S_1 = \{ y \in \widetilde{Y}: \|y\| \leq 1 \} \). Then it is easy to verify that \( F \) is strictly conically differentiable on \( R \) and

\[
D_x F(t)(1) = \{ \sum \alpha_n(t) 4n^2 y_n^2 \}; \quad D_x F(t)(-1) = -D_x F(t)(-1) = \{- \sum \alpha_n 4n^2 y_n^2 \}.
\]

We suppose that there exists a map \( g \) of \( R \) into \( Y \) such that
\[ F(t) = F(0) + g(t). \text{ Then } F(t) = F(0) + g(t) = S_1 + f(t). \]

Hence \( y = f(1) = g(1) \in Y \) and this contradicts the assumption \( y \in Y \).

References


