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FINITE GRAPHS AND DIGRAPHS WHICH ARE NOT RECONSTRUCTIBLE FROM THEIR CARDINALITY RESTRICTED SUBGRAPHS
Václav Nýdl

Abstract: For every \( n > 1 \) we construct two non-isomorphic graphs with \( 2n \) vertices having the same collection of \( n \)-vertex subgraphs. The constructions are given also for the case of disconnected, of unicyclic graphs and for trees. Finally the construction is modified so as to give examples of two non-isomorphic graphs with \( 3k+9 \) vertices having the same collection of \( 2k \)-vertex subgraphs.

Key words: Finite directed graphs, finite undirected graphs, Ulam conjecture.

Classification: 05C60

A digraph is a couple \( G = (V(G), E(G)) \) where \( V(G) \) is a set and \( E(G) \) an irreflexive binary relation on \( V(G) \). If the relation \( E(G) \) is symmetric (i.e., \( E(G) = E(G)^{-1} \)) then \( G \) is called a graph. For every digraph \( G \) its symmetrization \( \text{sym } G = (V(G), E(G) \cup E(G)^{-1}) \) is defined. For every subset \( Y \) of \( V(G) \) we denote \( G/Y = (Y, E(G) \cap Y^2) \). Now, for every natural \( k \) we define \( U_k(G) = \{ G/Y; Y \subseteq V(G), \text{card } Y = k \} \). We write \( G_1 \cong G_2 \) if there is a one-to-one correspondence \( \phi: U_k(G_1) \rightarrow U_k(G_2) \) such that for every \( H \in U_k(G_1) \) \( H \cong \phi(H) \) holds.

Proposition 1. For every \( n > 1 \) there are two digraphs \( G_1, G_2 \) such that \( \text{card } V(G_1) = \text{card } V(G_2) = 2n \), \( G_1 \not\cong G_2 \) and \( G_1 \cong G_2 \).

Proof: Let \( X = \{ x, y \} \cup \{ a_i; i=1, \ldots, n-1 \} \cup \{ b_i; i=1, \ldots \} \).
...,n-l$ be a set having 2n elements. We define the binary relation $R = \{ \langle a_i, a_{i+1} \rangle; i = 1, \ldots, n-2 \} \cup \{ \langle b_i, b_{i+1} \rangle; i = 1, \ldots, n-2 \}$.

Further let $R_1 = R \cup \{ \langle x, a_1 \rangle, \langle a_{n-1}, y \rangle \}$, $R_2 = R \cup \{ \langle x, a_1 \rangle, \langle b_{n-1}, y \rangle \}$. We have two disconnected digraphs $G_1 = \langle X, R_1 \rangle$, $G_2 = \langle X, R_2 \rangle$. The digraph $G_1$ has two components with $n+1$ and $n-1$ vertices, the digraph $G_2$ has two components both with $n$ vertices. Clearly $G_1$, $G_2$ are non-isomorphic.

Now, for every $i < n$ we define the set $Q(i) = \{ a_j; j=i, \ldots, n-1 \} \cup \{ b_j; j=i, \ldots, n-1 \} \cup \{ y \}$ and the set $\overline{Q}(i) = X - Q(i)$. For every $i=1, \ldots, n-1$ we define the bijection $f_i : X \rightarrow X$ as follows:

- $f_i(x) = x$, $f_i(y) = y$,
- $f_i(a_j) = a_j$, $f_i(b_j) = b_j$ for every $j < i$,
- $f_i(a_j) = b_j$, $f_i(b_j) = a_j$ for every $j \geq i$.

The restrictions of $f_i$ give isomorphisms $G_1/Q(i) \simeq G_2/Q(i)$, $G_1/\overline{Q}(i) \simeq G_2/\overline{Q}(i)$. Finally, we describe the mapping $\phi : U_n(G_1) \rightarrow U_n(G_2)$. If $Y$ is a subset of $X$ having $n$ elements then

- (a) $\phi(G_1/Y) = G_2/f_1(Y)$ for the case $x \notin Y$,
- (b) $\phi(G_1/Y) = G_2/Y$ for the case $x \in Y$, $y \notin Y$,
- (c) $\phi(G_1/Y) = G_2/f_k(Y)$ where $k = \min \{ i; a_i \notin Y, b_i \notin Y \}$ for the case $x \in Y$, $y \in Y$.

The existence of the number $k \leq n-1$ in (c) follows from the conditions $\text{card } Y = n$, $x \in Y$, $y \in Y$.

Further the mapping $\psi : U_n(G_2) \rightarrow U_n(G_1)$ can be defined, if we substitute the symbols $\psi$, $G_2$, $G_1$ for the symbols $\phi$, $G_1$, $G_2$ in (a), (b), (c). It follows immediately from the definitions of $f_i$, $\phi$, $\psi$ that for every $Y$ having $n$ elements it holds:
The analogous results can be obtained for undirected graphs applying the operation sym to \( G_1, G_2 \) from Proposition 1. The above described technique of the construction of \( \Phi \) can be used to prove some similar statements.

**Proposition 2.** For every \( n \geq 2 \) there are two oriented trees (and also two trees) \( T_1, T_2 \) such that \( \text{card } V(T_1) = \text{card } V(T_2) = 2n, T_1 \neq T_2, T_1 \sim_2 T_2. \)

Outline of proof: Let \( X = \{a_i; i=1,\ldots,n-1\} \cup \{x\} \) be a set having 2n elements, let \( R = \{\langle a_i, a_{i+1} \rangle; i=1,\ldots,2n-2\} \). Further, \( R_1 = R \cup \{\langle x, a_{n-1} \rangle\}, R_2 = R \cup \{\langle x, a_n \rangle\} \) and we take \( T_1 = \langle X, R_1 \rangle, T_2 = \langle X, R_2 \rangle. \) For the case of trees we take \( \text{sym } T_1, \text{sym } T_2. \)

**Proposition 3.** For every \( n > 2 \) there are two unicyclic digraphs (and also graphs) \( C_1, C_2 \) such that \( \text{card } V(C_1) = \text{card } V(C_2) = 2n, C_1 \neq C_2, C_1 \cong C_2. \)

Outline of proof: Let \( X = \{a_i; i=1,\ldots,n-1\} \cup \{x,y\} \) be a set having 2n elements, let \( R = \{\langle a_i, a_{i+1} \rangle; i=1,\ldots,n-2\} \cup \{\langle b_i, b_{i+1} \rangle; i=1,\ldots,n-2\} \) \cup \{\langle b_1, a_1 \rangle, \langle a_{n-1}, b_{n-1} \rangle\}. Further, \( R_1 = R \cup \{\langle x, a_1 \rangle, \langle a_{n-1}, y \rangle\}, R_2 = R \cup \{\langle x, b_1 \rangle, \langle a_{n-1}, y \rangle\}. \) We take \( C_1 = \langle X, R_1 \rangle, C_2 = \langle X, R_2 \rangle \) and for the case of undirected graphs \( \text{sym } C_1, \text{sym } C_2. \)

**Proposition 4.** For every \( k \geq 2 \) there are two graphs \( G_1, G_2 \) such that \( \text{card } V(G_1) = \text{card } V(G_2) = 3k+9, G_1 \cong G_2, G_1 \not\cong G_2. \)

Proof. Let \( k \geq 2 \) be given. Let \( M = \{a,b,c,1,2,3,11,12,13\} \)
be a set having 9 elements and let $N$ be the set of all natural numbers. We define $R = \{\{a, 2\}, \{2, b\}, \{b, 3\}, \{3, c\}, \{c, 1\}, \{1, a\}, \{a, 12\}, \{12, 2\}, \{b, 13\}, \{13, 3\}, \{c, 11\}, \{11, 1\}\}$. We denote $a_0 = a$, $b_0 = b$, $c_0 = c$ and for every $i \in N$ $a_i = \langle a, i \rangle$, $b_i = \langle b, i \rangle$, $c_i = \langle c, i \rangle$.

Now, for every $m \in N$ we define $A_m = \{a_i \mid 0 \leq i \leq m\}$, $B_m = \{b_i \mid 0 \leq i \leq m\}$, $C_m = \{c_i \mid 0 \leq i \leq m\}$, $R_A = \{\{a_i, a_{i+1}\} \mid 0 \leq i < m\}$, $R_B = \{\{b_i, b_{i+1}\} \mid 0 \leq i < m\}$, $R_C = \{\{c_i, c_{i+1}\} \mid 0 \leq i < m\}$. The graphs $G_1$, $G_2$ are defined as follows (see Fig. 1 for the case $k=5$):

$$
G_1 = (M \cup A_{k-1} \cup B_k \cup C_{k+1}, R \cup R_A \cup R_B \cup R_C),
$$
$$
G_2 = (M \cup A_{k+1} \cup B_k \cup C_{k-1}, R \cup R_A \cup R_B \cup R_C).
$$

Clearly, $\text{card } V(G_1) = \text{card } V(G_2) = 9 + (k-1) + k + (k+1) = 3k + 9$.

First, we prove that $G_1$, $G_2$ are non-isomorphic.

Let us suppose $\varphi : G_1 \rightarrow G_2$ is an isomorphism. Since $\varphi$ preserves degrees of vertices, it holds $\varphi(\{a, b, c\}) = \{a, b, c\}$, $\varphi(\{1, 2, 3\}) = \{1, 2, 3\}$, $\varphi(\{a_{k-1}, b_k, c_{k+1}\}) = \{a_{k+1}, b_k, c_{k-1}\}$. Thus, $\varphi(b_i) = b_i$ for every $i \neq k$, $\varphi(a_i) = c_i$ for every $i \neq k-1$, $\varphi(c_i) = a_i$ for every $i \neq k+1$. Further, we have $\varphi(2) = 3$, which yields $\{\varphi(12), 3\}$ and $\{\varphi(12), c\}$ are edges in $G_2$. No such a $\varphi(12)$ exists in $G_2$.

Secondly, we prove $G_1 \not\cong G_2$.

Let us denote $S_1 = \{G_1/Z \mid |Z| = 2k^3\}$, $S_2 = \{G_2/T \mid |T| = 2k^3\}$. We are going to describe the bijection $\hat{\varphi} : S_1 \rightarrow S_2$ such that for every $G \in S_1$ $\hat{\varphi}(G) \simeq G$.

We define the set $D = A_{k-1} \cup B_{k-1} \cup C_{k-1}$ and isomorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi$.
\[ \varphi_1 = \text{id}_{D \cup \{a_k^1\}} : D \cup \{a_k^1\} \to D \cup \{a_k^1\} \]

\[ \varphi_2 : D \cup \{a_k, c_{k+1}^1\} \to D \cup \{a_k, c_{k+1}^1\}, \text{ where } \varphi_2(1) = 2, \varphi_2(2) = 3, \varphi_2(3) = 1, \varphi_2(11) = 12, \varphi_2(12) = 13, \varphi_2(13) = 11, \varphi_2(a_i) = b_i, \varphi_2(b_i) = c_i, \varphi_2(c_i) = a_i. \]

\[ \varphi_3 : D \to D, \text{ where } \varphi_3(x) = \varphi_3^{-1}(x) \text{ for every } x \in D. \]

\[ \psi : D \cup A_{k-1} \cup B_k \cup C_{k+1} \to D \cup A_{k+1} \cup B_k \cup C_{k-1}, \text{ where } \psi(a_i) = c_i, \psi(b_i) = b_i, \psi(c_i) = a_i. \]

Let \( G = G_1/Z \in S_1. \)

I. If \( D \cap b_k, c_k, c_{k+1}^1 = \emptyset \) then \( \Phi(C) = G_2/\psi_1(Z) = G_2/Z. \)

II. If \( D \cap b_k, c_k, c_{k+1}^1 \neq \emptyset \) then we discuss 4 conditions:

1) \( Z \) satisfies \((\alpha)\). We define \( i_0 = \min \{i \mid D \cap a_i, b_i, c_i^1 = \emptyset \} \) and \( Z_1 = Z \cap \{a_i \mid i > i_0\} \cup \{b_i \mid i > i_0\} \cup \{c_i \mid i > i_0\} \). Let \( Z_2 = Z - Z_1 \).

Then \( \Phi(G) = G_2/\varphi_1(Z_2) \cup \psi(Z_1). \)

2) \( Z \) does not satisfy \((\alpha)\) and \( Z \) satisfies \((\beta)\). We define \( i_0 = \min \{i \mid D \cap a_i, b_i, c_i^1 \} \) and \( Z_1 = Z \cap \{b_i \mid i > i_0\} \cup \{c_i \mid i > i_0\} \). Let \( Z_2 = Z - Z_1 \).

Then \( \Phi(G) = G_2/\varphi_3(Z_2) \cup \psi(Z_1). \)

3) \( Z \) does not satisfy \((\alpha),(\beta)\) and \( Z \) satisfies \((\gamma)\). We define \( i_0 = \min \{i \mid D \cap a_i, b_i, c_i^1 \} \) and \( Z_1 = Z \cap \{a_i \mid i > i_0\} \cup \{c_i \mid i > i_0\} \). Let \( Z_2 = Z - Z_1 \).

Then \( \Phi(G) = G_2/\varphi_1(Z_2) \cup \psi(Z_1). \)

4) \( Z \) does not satisfy \((\alpha),(\beta),(\gamma)\) and \( Z \) satisfies \((\delta)\). 

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We define \( i_0 = \min \{ i \mid Z \cap \{a_i, b_i, c_i \} = \{c_i \} \} \) and \( Z_1 = Z \cap \{a_i \mid i > i_0 \} \cup \{c_i \mid i > i_0 \} \), \( Z_2 = Z - Z_1 \).

Then \( \Phi(G) = G_2/\varphi_2(Z_2) \cup \varphi(Z_1) \).

5) Let us suppose \( Z \) does not satisfy \((\alpha), (\beta), (\gamma), (\delta')\). Then for every \( i, 0 \leq i \leq k-1 \), \( |Z \cap \{a_i, b_i, c_i \} \| \geq 2 \). Thus, \( |Z| \geq 2k+1 \). \( G_1/Z \notin S_1 \).

It can be easily shown (using the method of discussion again) that for every \( G_2/T \in S_2 \) there exists \( G_1/Z \in S_1 \) such that \( G_2/T = \Phi(G_1/Z) \). So, \( \Phi \) is fully determined and has all the needed properties.

Fig. 1
References


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