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Commentationes Mathematicae Universitatis Carolinae, Vol. 22 (1981), No. 2, 305--309

Persistent URL: <http://dml.cz/dmlcz/106076>

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ON EXTENDING TRANSITIVE HOMEOMORPHISMS FROM
THE CANTOR SET TO THE PRODUCT OF TWO CANTOR SETS

Ryszard FRANKIEWICZ and Andrzej GUTEK

Abstract: We prove the following theorem: Let f be a transitive homeomorphism from the Cantor set C onto itself. Then there exists a homeomorphism g from C onto itself such that $g(x) = x$ for some point x of C and the homeomorphism $g \times f: C \times C \rightarrow C \times C$ is transitive. More precisely, if the semiorbit $\{f^n(y): n=1,2,\dots\}$ is dense in C , then the homeomorphism g can be defined in such a way that for some point z of C the semiorbit $\{g^n(z), f^n(y): n=1,2,\dots\}$ is dense in $C \times C$.

Key words and phrases: Transitive homeomorphism, Cantor set, cartesian product.

Classification: 54C10, 54C20.

There is a number of papers in which possibilities of extending homeomorphisms are investigated. One can list papers of Knaster and Reichbach [3], J. Pollard [7], R.S. Pierce [6], J.W. Baker [1] and J. van Mill [4]. A possibility of extension to a transitive homeomorphism is studied in [2].

Let us remind that a homeomorphism h from the space X onto itself is said to be transitive if and only if there exists a point x whose orbit $\{h^n(x): n \text{ is an integer}\}$ is dense in X . We shall use the following property of transitive homeomorphisms:

Lemma (Oxtoby [5], p. 70). Let X be a complete, sepa-

rable metric space without isolated points, and let h be a transitive homeomorphism from X onto itself. Then those points whose positive semiorbit is dense constitute a residual set in X .

A set is said to be residual iff it is the complement of a set of first category. A positive semiorbit of a point x is the set $\{h^n(x): n=1,2,\dots\}$.

Theorem. Let f be a transitive homeomorphism from the Cantor set C onto itself. Then there exists a homeomorphism g from C onto itself such that $g(x) = x$ for some point x of C and the homeomorphism $g \times f: C \times C \rightarrow C \times C$ is transitive. More precisely, if the semiorbit $\{f^n(y): n=1,2,\dots\}$ is dense in C , then the homeomorphism g can be defined in such a way that for some point z of C the semiorbit $\{g^n(z), f^n(y): n=1,2,\dots\}$ is dense in $C \times C$.

Proof. Let us assume that the Cantor set C is given by the usual ternary expansion, and let \mathcal{B} denote the basis defined by this expansion, i.e. it is a family of closed-open subsets of C , and if $I, J \in \mathcal{B}$, then $I \subseteq J$ or $J \subseteq I$ or $I \cap J = \emptyset$ and $\text{diam } I = 3^i \cdot \text{diam } J$ for some integer i .

For every two subsets A and B of C put $A < B$ if and only if $a < b$ for every $a \in A$ and $b \in B$. For every set B of the basis \mathcal{B} and for every positive integer k consider a partition $\{B(m,k): m=1,\dots,k\}$ of B into k disjoint subsets belonging to \mathcal{B} diameters of which are less or equal to $k^{-1} \cdot \text{diam } B$ and such that $B(m,k) < B(p,k)$ for $m < p \leq k$. If k is equal to 2^j for some positive integer j , then we require diameters of any two such subsets to be equal one to another.

Let $B_1, B_2, \dots, B_n, \dots$ be defined by $B_n = [\frac{2}{3^n}, \frac{1}{3^{n-1}}] \cap C$.
 Let the semiorbit $\{f^n(y): n=1, 2, \dots\}$ of a point y be dense in C . Such a point exists in virtue of the lemma.

The homeomorphism g is defined by induction.

The first step. g_1 is the linear and order preserving mapping from $B_k(1, 2)$ onto $B_1(2, 2)$.

Put z equal to $\frac{2}{3}$ and consider the point $\langle \frac{2}{3}, y \rangle$.

Thus we have obtained a chain consisting of the sets $A_1^1 = B_1(1, 2)$ and $A_2^1 = B_1(2, 2)$, and $g_1(A_1^1(m, 2^k)) = A_2^1(m, 2^k)$ for every positive integer k and for $m=1, \dots, 2^k$.

The n-th step ($n \geq 2$). Suppose we have constructed a chain $A_1^{n-1}, \dots, A_k^{n-1}$ of closed-open segments of C such that $\cup \{A_j^{n-1}: j=1, \dots, k\} = \cup \{B_j: j=1, \dots, n-1\}$ and $A_1^{n-1}, A_k^{n-1} \subseteq B_{n-1}$, and a function g_{n-1} defined on $\cup \{A_j^{n-1}: j=1, \dots, k-1\}$ such that $g_{n-1}|_{A_j^{n-1}}$ is linear and order preserving mapping from A_j^{n-1} onto A_{j+1}^{n-1} for $j=1, \dots, k-1$. For each segment A_j^{n-1} let us consider cartesian products $P_j^{n-1}(t, i) = A_j^{n-1} \times C(t, 2^{n-i})$, where $t=1, \dots, 2^{n-i}$, and i is such a positive integer that $A_j^{n-1} \subseteq B_i$. Let N_n be the number of sets $P_j^{n-1}(t, i)$, where $i=1, \dots, n-1$ and $j=1, \dots, k$ and $t=1, \dots, 2^{n-i}$. Order these sets, putting the first one this set, which contains the point $\langle \frac{2}{3}, y \rangle$. Denote the s -th set in this ordering by P_s , and put $n^+(P_s) = k-j$ and $n^-(P_s) = j$ if and only if $P_s = P_j^{n-1}(t, i)$ for some j, t and i . We define for each set P_s numbers n_s and \bar{m}_s . We put $n_1 = 0$ and $\bar{m}_1 = 1$. Suppose we have defined n_r and \bar{m}_r for $r < s$. We put n_s to be such a positive integer that $n_s - n_{s-1} - n^-(P_s) - n^+(P_{s-1}) = \bar{m}_s$ is a positive integer, and such that

$f^{n_s}(y) \in C(t, 2^{n-i})$, where t and i are such that $P_s = A_j^{n-1} \times C(t, 2^{n-i})$. Such a positive integer n_s exists, because the positive semiorbit of the point y is dense in C .

Put $M_n = \bar{m}_1 + \bar{m}_2 + \dots + \bar{m}_{N_n} + 1$. Let us consider a partition $\{B_n(m, M_n) : m=1, \dots, M_n\}$ of B_n into M_n disjoint subsets belonging to \mathcal{B} diameters of which are less or equal to $M_n^{-1} \cdot \text{diam } B_n$. Let the sets $A_1^{n-1}(s, N_n)$, where $s=1, \dots, N_n$, constitute the similar partition of A_1^{n-1} into N_n disjoint and non-void subsets. This partition induces partitions of the sets A_j^{n-1} , where $j=2, \dots, k$, if we put $A_j^{n-1}(s, N_n) = g_{n-1}^{j-1}(A_1^{n-1}(s, N_n))$.

Observe that if $\frac{2}{j} \in A_j^{n-1}$, then $\frac{2}{j} \in A_j^{n-1}(1, N_n)$. Let $m_s = \bar{m}_1 + \dots + \bar{m}_s$ for $s=1, \dots, N_n$.

We define g_n as follows:

$g_n|_{B_n(m, M_n)}$ is a linear and order preserving mapping from $B_n(m, M_n)$ onto $A_1^{n-1}(s, N_n)$ iff $m = m_s$ for some s , and onto $B_n(m+1, N_n)$ otherwise,

$g_n|_{A_k^{n-1}(s, N_n)}$ is a linear and order preserving mapping from $A_k^{n-1}(s, N_n)$ onto $B_n(m_{s+1}, M_n)$, where $m_{N_n+1} = M_n$ and

$$g_n|_{\cup \{B_j : j=1, \dots, n-1\} \setminus A_k^{n-1}} = g_{n-1}.$$

Put $A_1^n = B_n(1, M_n)$ and $A_{j+1}^n = g_n(A_j^n)$ for $j=1, \dots, M_n+k N_n-1$.

Thus, g_n is defined on $\cup \{B_j : j=1, \dots, n\} \setminus B_n(M_n, M_n)$ and is continuous and one-to-one. Let us observe that

$(g_n \times f)^{n_s} \langle \frac{2}{j}, y \rangle \in P_s$ for $s=1, \dots, N_n$, and $A_1^n, A_{M_n+k N_n}^n \subseteq B_n$. The mapping $g: C \rightarrow C$ defined by $g|_{B_n} = g_{n+1}|_{B_n}$ and $g(0) = 0$ is a homeomorphism from C onto itself, and the positive semiorbit

of the point $\langle \frac{2}{3}, y \rangle$ is dense in $C \times C$.

R e f e r e n c e s

- [1] J. W. BAKER: Extensions of homeomorphisms over metric spaces, preprint.
- [2] A. GUTEK: On extending homeomorphisms on the Cantor set, Topological Structures II, Mathematical Centre Tracts 115(1979), 105-116.
- [3] B. KNASTER and M. REICHBACH: Notion d'homogénéité et prolongements des homéomorphies, Fund. Math. 40(1953), 180-193.
- [4] J. van MILL: Characterization of some zero-dimensional separable metric spaces, preprint.
- [5] J.C. OXTOBY: Measure and category, Springer Verlag, New York-Heidelberg-Berlin 1971.
- [6] R.S. PIERCE: Existence and uniqueness theorems for extensions of zero-dimensional compact metric spaces, Trans. Amer. Math. Soc. 148(1970), 1-21.
- [7] J. POLLARD: On extending homeomorphisms on zero-dimensional spaces, Fund Math. 67(1970), 39-48.

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(Oblatum 8.9. 1980)